A note on a conjecture of Calderón*

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Abstract

For $f \in \mathcal{S}(R^2)$ and $\Omega \in L^1(S^1)$, $\int_{S^1} \Omega(x')dx' = 0$, define

$$T_\Omega(f)(x) = \lim_{\epsilon \to 0+} \int_{|x-y| \geq \epsilon} \frac{\Omega(y/|y|)}{|y|^2} f(x-y)dy.$$

In this paper, we shall prove that there are a class of functions in $H^1(S^1) - L\ln^+ L(S^1)$ such that T_Ω is weak type L^1-bounded.

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1 Introduction

For $f \in \mathcal{S}(R^d)$ and $\Omega \in L^1(S^{d-1})$, $\int_{S^{d-1}} \Omega(x')dx' = 0$, define

$$T_\Omega(f)(x) = \lim_{\epsilon \to 0+} \int_{|x-y| \geq \epsilon} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)dy. \tag{1}$$

In [1], Calderón and Zygmund proved that if $\Omega \in L\ln^+ L(S^{d-1})$, i.e.

$$\int_{S^{d-1}} |\Omega(x')| \ln(2 + |\Omega(x')|)dx' < \infty, \tag{2}$$

T_Ω is L^p-bounded for $1 < p < \infty$. In [7] and [5], Ricci, Weiss and independently Connett proved that if $\Omega \in H^1(S^{d-1})$, T_Ω is L^p-bounded for $1 < p < \infty$. Also see [4]. But, it remained open for long time if T_Ω is weak type L^1-bounded under the correspongding conditions (for $\Omega \in L\ln^+ L(S^{d-1})$, it is a conjecture of Calderón).

In [3], Christ and Rubio de Francia proved that for $\Omega \in L\ln^+ L(S^{d-1})$ $(d \leq 7)$, T_Ω is weak type L^1-bounded. And at the same time, in [6], Hofmann independently proved that for $\Omega \in L^q(S^{d-1})$ $(d = 2, q > 1)$, T_Ω is weak type L^1-bounded. Finally, in [8], Seeger generalized the result to all $d \geq 2$ and $\Omega \in L\ln^+ L(S^{d-1})$.

We know that $L\ln^+ L(S^{d-1}) \subset H^1(S^{d-1})$. So, it is natural to ask a harder question: for $\Omega \in H^1(S^{d-1})$, is T_Ω weak type L^1-bounded? In [9], Stefanov proved that if Ω is a finite sum of $H^1(S^1)-$atoms with the additional assumption that the atoms are supported on almost disjoint arcs of comparable size (note that such an Ω must be $L^\infty(S^1)-$function), $\|T_\Omega\|_{L^1 \to WL^1}$ essentially depends only on $\|\Omega\|_{H^1}$. Precisely, he proved

Theorem 1 *Let N, l be positive integers satisfying $N \leq 2\pi 2^l c_0$ where c_0 is suitably choosen, and I_n denote the arc in S^1 with center e_n and satisfying $|I_n| \sim 2^{-l}$, $|I_n \cap I_m| \leq \frac{1}{2}\min(|I_n|, |I_m|)$ for $n \neq m$. Suppose $\Omega = \sum_1^N \lambda_n a_n$ where $\lambda_n > 0$ and a_n is an $H^1(S^1)-$atom on S^1 satisfying supp$(a_n) \subset I_n$, $\|a_n\|_\infty \leq 2^{-l}$ and $\int_{S^1} a_n(\theta)d\theta = 0$. Then, $\|T_\Omega\|_{L^1 \to WL^1} \leq C \cdot (\sum_{n=1}^N \lambda_n)$ where C is independent of N and l.*

In this paper, we shall prove that there are a class of functions in $H^1(S^1) - L\ln^+ L(S^1)$ such that T_Ω is weak type L^1-bounded. We have

Theorem 2 *Let I_n denote the arc in S^1 with center e_n and length $2\rho_n$, disjoint mutually, and*

$$A = \sup_{\theta \in S^1} \sum_{n:\theta \notin I_n} \frac{\rho_n^2}{|\theta - e_n|^2} < \infty; \tag{3}$$

let $\Omega = \sum_1^\infty \lambda_n a_n$ where $\sum_1^\infty |\lambda_n| < \infty$ and a_n is an $H^1(S^1)-$atom on S^1 satisfying

$$\begin{array}{ll} (i) & supp(a_n) \subset I_n, \\ (ii) & \int_{S^1} a_n(\theta) d\theta = 0, \\ (iii) & \|a_n\|_\infty \leq \rho_n^{-1}, \end{array} \tag{4}$$

Then,

$$|\{x : |T_\Omega(f)(x)| > \lambda\}| \leq C \cdot A(\sum_n \lambda_n)\lambda^{-1} \|f\|_1 \tag{5}$$

for all $f \in \mathcal{S}(R^2)$ and $\lambda > 0$, where C is independent of f, λ and Ω.

Theorem 3 *Suppose $\rho_n \in (0,1)$ and $\sum_n \rho_n < \infty$. There are $\{e_n\} \subset S^1$, arcs $I_n \subset S^1$ with center e_n and length $2\rho_n$ such that for any sequence of H^1-atoms $\{a_n\}$ (satisfying (4))and $\Omega = \sum_n \lambda_n a_n$,*

$$|\{x : |T_\Omega(f)(x)| > \lambda\}| \leq C \cdot \sum_n \rho_n \cdot \sum_n \lambda_n \cdot \|f\|_1 \tag{6}$$

for all $f \in \mathcal{S}(R^2)$ and $\lambda > 0$, where C is independent of f, λ and Ω.

From Theorem 3, we have

Corollary 4 *For any increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0+) = 0$, if*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \tag{7}$$

then, there must be an $\Omega \in H^1 - \varphi(L)$ such that T_Ω is weak type L^1-bounded, where

$$\varphi(L) = \left\{\Omega : \int_{S^1} \varphi(|\Omega(\theta)|) d\theta < \infty\right\}.$$

2 Some lemmas

Without loss of generality, we always assume that $\sum_1^\infty \lambda_n = 1$ and $\lambda_n > 0$.

For a rectangle $Q = Q(y, \mathbf{r})(\subset R^2)$ with center y and sides' length $2\mathbf{r} = (2r_1, 2r_2)$, let $mQ = Q(y, m\mathbf{r})$, $\overline{d}(Q) = \max(r_1, r_2)$. Set

$$\mathcal{A}_n = \left\{ Q : \begin{array}{c} \text{the longer side of } Q \text{ is parallel to } e_n, \text{ the shorter} \\ \text{side length is } \rho_n \text{ time of the longer side length} \end{array} \right\},$$

and

$$M_n(f)(x) = \sup_{x \in Q \in \mathcal{A}_n} \frac{1}{|Q|} \int_Q |f(z)|\, dz.$$

It is easy to see that M_n is weak type L^1-bounded and $\sup_n \|M_n\|_{L^1 \to WL^1} < \infty$. Now, we first give a modified Whitney's decomposition.

Lemma 5 *Suppose $E \subset R^2$ is open. There is $m \in R^+$ such that for any n, there are mutually disjoint rectangles $\{Q_{n,i}\} \subset \mathcal{A}_n$ satisfying*

$$\begin{array}{ll} (i) & E = \cup_i Q_{n,i} \\ (ii) & 4Q_{n,i} \subset E \\ (iii) & mQ_{n,i} \cap E^c \neq \emptyset. \\ (iv) & \overline{d}(Q_{n,i}) \in \left\{ 2^k : k = 0, \pm 1, \pm 2, \cdots \right\}. \end{array}$$

This lemma can be proved along the idea of the proof of the Whitney's decomposition, see[10].

For $f \in L^1(R^2)$ and $\lambda > 0$, let

$$\begin{array}{l} E_n = \left\{ x : M_n(f)(x) > \frac{\lambda}{\lambda_n} \right\} \\ E = \{x : M(f)(x) > \lambda\} \cup (\cup_n E_n) \end{array}$$

where M is the Hardy-Littlewood maximal operator. For any n, we shall make $C - Z$ decomposition of f based on the modified Whitney's decomposition of E (not E_n, key point). By Lemma 5, we have that $E = \cup_i Q_{n,i}$ where $\{Q_{n,i}\}$ satisfy the conditions in Lemma 5. Take

$$\begin{array}{l} b_{Q_{n,i}} = (f(x) - \frac{1}{|Q_{n,i}|} \int_{Q_{n,i}} f(y)dy)\chi_{Q_{n,i}}(x) \\ g_n(x) = \sum_i \frac{1}{|Q_{n,i}|} \int_{Q_{n,i}} f(y)dy \chi_{Q_{n,i}}(x) \\ g(x) = f(x)\chi_{E^c}(x) \\ B_{n,j} = \sum_{i : \overline{d}(Q_{n,i}) = 2^j} b_{Q_{n,i}}. \end{array}$$

We have

Lemma 6 *For any n, $f = g + g_n + \sum_i b_{Q_{n,i}}$, and*

$$\begin{array}{ll} (i) & \int_{R^2} b_{Q_{n,i}}(x)dx = 0 \\ & \int_{R^2} \left| b_{Q_{n,i}}(x) \right| dx \leq C \cdot \frac{\lambda}{\lambda_n} \left| Q_{n,i} \right| \\ & \sum_i \int_{Q_{n,i}} \left| b_{Q_{n,i}}(x) \right| dx \leq 2 \left\| f \right\|_1 \\ (ii) & \left\| g \right\|_2^2 \leq C\lambda \left\| f \right\|_1, \left\| g_n \right\|_2^2 \leq C \cdot \frac{\lambda}{\lambda_n} \left\| f \right\|_1 \\ (iii) & |E| \leq \sum_n |E_n| + |\{x : M(f)(x) > \lambda\}| \leq C \cdot \lambda^{-1} \left\| f \right\|_1 . \end{array}$$

Lemma 7 *Under the hypothesis of Theorem 2,*

$$\left\| \sum_n \lambda_n \frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} * g_n \right\|_2^2 \leq C \cdot A \cdot \lambda \left\| f \right\|_1 \tag{8}$$

for all $f \in \mathcal{S}(R^2)$ and $\lambda > 0$.

Proof. We first prove that

$$\sum_n \left| \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} \right)^\wedge (\xi) \right|^2 \leq C \cdot A. \tag{9}$$

By a well-known computation (see [10] page 39),

$$\left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} \right)^\wedge (\xi) = \int_{S^1} a_n(\theta) \left(\ln \frac{1}{\left| \langle \theta, \frac{\xi}{|\xi|} \rangle \right|} + \frac{\pi i}{2} sign \langle \theta, \xi \rangle \right) d\theta \tag{10}$$

which is independent of $|\xi|$. So, we may assume $|\xi| = 1$. Let $\xi^\perp \in S^1$ denote anyone of the TWO unit vectors orthogonal to ξ. If $\pm\xi^\perp \notin I_n$, $sign\langle \theta, \xi \rangle$ is constant for $\theta \in I_n$, thus (without loss of generality, we may assume that $\left| \angle(e_n, \xi^\perp) \right| < \pi/2$, in this case, $|\langle e_n, \xi \rangle| \sim \left| e_n - \xi^\perp \right|$)

$$\begin{aligned} \left| \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} \right)^\wedge (\xi) \right| &= \left| \int_{S^1} a_n(\theta) \left(\ln |\langle \theta, \xi \rangle|^{-1} - \ln |\langle e_n, \xi \rangle|^{-1} \right) d\theta \right| \\ &\leq \int_{S^1} |a_n(\theta)| \ln(1 + |\langle \theta - e_n, \xi \rangle| \, |\langle e_n, \xi \rangle|^{-1}) d\theta \\ &\leq C \sup_{\theta \in I_n} \frac{|\theta - e_n|}{\left| e_n - \xi^\perp \right|} \leq C \frac{\rho_n}{\left| e_n - \xi^\perp \right|}. \end{aligned} \tag{11}$$

By (11), we have (note that $A \geq 1$)

$$\begin{aligned} \sum_n \left| \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} \right)^\wedge (\xi) \right|^2 &= \left(\sum_{n:\xi^\perp \in I_n \text{ or } -\xi^\perp \in I_n} + \sum_{n:\pm\xi^\perp \notin I_n} \right) \left| \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} \right)^\wedge (\xi) \right|^2 \\ &\leq C(1 + \sum_n \frac{\rho_n^2}{\left| e_n - \xi^\perp \right|^2}) \leq C' \cdot A. \end{aligned} \tag{12}$$

Thus

$$\left\|\sum_n \lambda_n \frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2} * g_n\right\|_2^2 = \left\|\sum_n \lambda_n \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}\right)^\wedge (g_n)^\wedge\right\|_2^2$$

$$= \int_{S^1} \sum_n \sum_m \lambda_n \left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}\right)^\wedge (\xi)\,(g_n)^\wedge(\xi)\lambda_m \left(\frac{a_m(\frac{\cdot}{|\cdot|})}{|\cdot|^2}\right)^\wedge (\xi)\,(g_m)^\wedge(\xi)d\xi$$

$$\le \int_{S^1} \sum_m \lambda_m^2 \left|(g_m)^\wedge(\xi)\right|^2 \cdot \sum_n \left|\left(\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}\right)^\wedge (\xi)\right|^2 d\xi$$

$$\le C\cdot A\sum_m \lambda_m^2 \|\widehat{g_m}\|_2^2 \le C\cdot A\sum_m \left(\lambda_m^2 \cdot \frac{\lambda}{\lambda_m}\|f\|_1\right) \le C\cdot A\|f\|_1 .$$

Lemma 7 is proved.

Take $\beta \in C_c^\infty((\frac{1}{2},2))$ such that $0 \le \beta \le 1$ and $\sum_j \beta(2^{-j}t) = 1$ for all $t \in R^+$, $\psi \in C_c^\infty((-1,1))$ such that $0 \le \psi \le 1$ and $\psi|_{[-\frac{1}{2},\frac{1}{2}]} = 1$ and $\psi^{(k)}(\pm\frac{1}{2}) = 0$ for all $k = 0,1,2,\cdots$. Let

$$\begin{aligned} a_{n,j}(x) &= \beta(2^{-j}|x|)\frac{a_n(x/|x|)}{|x|^2} \\ \varphi_n^s(x) &= \left(\psi(2^{-\frac{s}{6}}\rho_n^{-1}\langle\tfrac{\cdot}{|\cdot|},e_n\rangle)\right)^\vee(x). \end{aligned} \tag{13}$$

We have

Lemma 8 *Under the hypothesis of Theorem 2,*

$$\left\|\sum_{s>0}\sum_n\sum_j \lambda_n a_{n,j} * \varphi_n^s * B_{n,j-s}\right\|_2^2 \le C\cdot A\lambda\|f\|_1 . \tag{14}$$

Proof. By a similar estimate in [9] (see section 5 below for details), we have

$$\left\|\sum_j a_{n,j} * B_{n,j-s}\right\|_2^2 \le C2^{-\frac{s}{2}}\frac{\lambda}{\lambda_n}\|f\|_1 . \tag{15}$$

Now, we estimate $\sum_n \left|\widehat{\varphi_n^s}(\xi)\right|^2$ where $s \ge 0$. If $\widehat{\varphi_n^s}(\xi) = \psi(2^{-\frac{s}{6}}\rho_n^{-1}\langle\frac{\xi}{|\xi|},e_n\rangle) > 0$, we have $\left|2^{-\frac{s}{6}}\rho_n^{-1}\langle\frac{\xi}{|\xi|},e_n\rangle\right| \le 1$, thus

$$\left|\langle\frac{\xi}{|\xi|},e_n\rangle\right| \le 2^{\frac{s}{6}}\rho_n. \tag{16}$$

Note that $\left|\langle\frac{\xi}{|\xi|},e_n\rangle\right| \sim \left|\xi^\perp - e_n\right|$ for all n satisfying $\pm\xi^\perp \notin I_n$ and $\left|\angle(e_n,\xi^\perp)\right| < \pi/2$, we have $\left|\xi^\perp - e_n\right| \le C2^{\frac{s}{6}}\rho_n$, and thus $\rho_n\left|\xi^\perp - e_n\right|^{-1} \ge C2^{-\frac{s}{6}}$. By Theorem 2 (3), the number of n satisfying (16) does not exceed $CA2^{\frac{s}{3}}$, thus

$$\sum_n \left|\widehat{\varphi_n^s}(\xi)\right|^2 \le CA2^{\frac{s}{3}}. \tag{17}$$

By (15) and (17),

$$\begin{aligned}\left\|\sum_{s>0}\sum_n\sum_j \lambda_n a_{n,j}*\varphi_n^s*B_{n,j-s}\right\|_2^2 &\le \left(\sum_{s>0}\left\|\sum_n\sum_j \lambda_n a_{n,j}*\varphi_n^s*B_{n,j-s}\right\|_2\right)^2\\ &= \left(\sum_{s>0}\left\|\sum_n \widehat{\varphi_n^s}\left(\lambda_n\sum_j \widehat{a_{n,j}}\widehat{B_{n,j-s}}\right)\right\|_2\right)^2\\ &\le \left(\sum_{s>0}\left\|\left(\sum_n\left|\widehat{\varphi_n^s}(\xi)\right|^2\right)^{1/2}\left(\sum_n\lambda_n^2\left(\sum_j\widehat{a_{n,j}}\widehat{B_{n,j-s}}\right)^2\right)^{1/2}\right\|_2\right)^2\\ &\le \left(\sum_{s>0}\left(CA2^{\frac{s}{3}}\sum_n\lambda_n^2\left\|\sum_j a_{n,j}*B_{n,j-s}\right\|_2^2\right)^{1/2}\right)^2\\ &\le \left(\sum_{s>0}\left(CA2^{\frac{s}{3}}2^{-\frac{s}{2}}\lambda\left(\sum_n\lambda_n\right)\|f\|_1\right)^{1/2}\right)^2 = CA\lambda\|f\|_1.\end{aligned}$$

Lemma 8 is proved.

Lemma 9 *Under the hypothesis of Theorem 2,*

$$\sum_{s>0}\sum_n\sum_j\lambda_n\left\|a_{n,j}*(\delta-\varphi_n^s)*B_{n,j-s}\right\|_1\le C\|f\|_1. \tag{18}$$

Proof. By a similar estimate in [9] (see section 5 below for details), we have

$$\left\|a_{n,j}*(\delta-\varphi_n^s)\right\|_1\le C2^{-\frac{s}{12}}. \tag{19}$$

So

$$\begin{aligned}&\sum_{s>0}\sum_n\sum_j\lambda_n\left\|a_{n,j}*(\delta-\varphi_n^s)*B_{n,j-s}\right\|_1\\ &\qquad\le\sum_{s>0}\sum_n\sum_j\lambda_n C2^{-\frac{s}{12}}\left\|B_{n,j-s}\right\|_1 = C\|f\|_1.\end{aligned}$$

Lemma 9 is proved.

3 Proof of Theorem 2

For fixed f and λ,

$$T_\Omega(f)(x)=T_\Omega(g)(x)+\sum_n\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*g_n(x)+\sum_n\sum_i\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*b_{Q_{n,i}}(x),$$

so,

$$
\begin{aligned}
&|\{x:|T_\Omega(f)(x)|>\lambda\}|\le I+II+III\\
&I=|\{x:|T_\Omega(g)(x)|>\lambda/3\}|\\
&II=\left|\left\{x:\left|\sum_n\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*g_n(x)\right|>\lambda/3\right\}\right|\\
&III=\left|\left\{x:\left|\sum_n\sum_i\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*b_{Q_{n,i}}(x)\right|>\lambda/3\right\}\right|.
\end{aligned}
\tag{20}
$$

By L^2-boundedness of T_Ω,

$$
\begin{aligned}
I &\le C\lambda^{-2}\|T_\Omega(g)\|_2^2\le C\lambda^{-2}\|\Omega\|_{H^1}^2\|g\|_2^2\\
&\le C\lambda^{-2}\left(\sum_n\lambda_n\right)^2\lambda\|f\|_1=C\lambda^{-1}\|f\|_1.
\end{aligned}
\tag{21}
$$

Similarly, we have

$$
II\le C\lambda^{-2}\left\|\sum_n\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*g_n(x)\right\|_2^2\le CA\lambda^{-1}\|f\|_1.
\tag{22}
$$

For III, we have

$$
\begin{aligned}
\sum_n\sum_i\lambda_n\frac{a_n(\frac{\cdot}{|\cdot|})}{|\cdot|^2}*b_{Q_{n,i}}(x)=&\sum_{s\le0}\sum_j\sum_n\lambda_n a_{n,j}*B_{n,j-s}(x)\\
&+\sum_{s>0}\sum_j\sum_n\lambda_n a_{n,j}*B_{n,j-s}(x)\\
=&\sum_{s\le0}\sum_j\sum_n\lambda_n a_{n,j}*B_{n,j-s}(x)\\
&+\sum_{s>0}\sum_j\sum_n\lambda_n a_{n,j}*\varphi_n^s*B_{n,j-s}(x)\\
&+\sum_{s>0}\sum_j\sum_n\lambda_n a_{n,j}*(\delta-\varphi_n^s)*B_{n,j-s}(x)
\end{aligned}
\tag{23}
$$

If $\overline{d}(Q_{n,i})\ge2^j$, $supp(a_{n,j}*b_{Q_{n,i}})\subset4Q_{n,i}$, so

$$
supp(\sum_{s\le0}\sum_j\sum_n\lambda_n a_{n,j}*B_{n,j-s})\subset\cup_n\cup_i4Q_{n,i}=E.
$$

Noticing that $|E|\le C\lambda^{-1}\|f\|_1$, we have

$$
\begin{aligned}
III\le\ &|E|+\left|\left\{x:\left|\sum_{s>0}\sum_j\sum_n\lambda_n a_{n,j}*\varphi_n^s*B_{n,j-s}(x)\right|>\lambda/9\right\}\right|\\
&+\left|\left\{x:\left|\sum_{s>0}\sum_j\sum_n\lambda_n a_{n,j}*(\delta-\varphi_n^s)*B_{n,j-s}(x)\right|>\lambda/9\right\}\right|\\
=\ &|E|+IV+V.
\end{aligned}
\tag{24}
$$

By Lemma 8,

$$IV \leq C\lambda^{-2} \left\| \sum_{s>0} \sum_j \sum_n \lambda_n a_{n,j} * \varphi_n^s * B_{n,j-s} \right\|_2^2 \leq CA\lambda^{-1} \|f\|_1 . \tag{25}$$

By Lemma 9,

$$V \leq C\lambda^{-1} \left\| \sum_{s>0} \sum_j \sum_n \lambda_n a_{n,j} * (\delta - \varphi_n^s) * B_{n,j-s} \right\|_1 \leq CA\lambda^{-1} \|f\|_1 . \tag{26}$$

Combining (20)-(22) and (24)-(26), we get

$$|\{x : |T_\Omega(f)(x)| > \lambda\}| \leq CA\lambda^{-1} \|f\|_1 .$$

Theorem 2 is proved.

4 Proof of Theorem 3

Without loss of generality, we may assume that $\{\rho_n\}$ is decreasing and $\sum_n \rho_n < \pi/64$. Let $d_m = \frac{1}{m^2} \sup_{n \leq m} n^2 \rho_n$. We have

Lemma 10 $\sum_m d_m \leq 16 \sum_n \rho_n$.

Proof. For $n \leq m$, $n^2\rho_n \leq n^2(n^{-1}\sum_{i\leq n}\sqrt{\rho_i})^2 \leq (\sum_{i\leq m}\sqrt{\rho_i})^2$, thus

$$\sum_m d_m \leq \sum_m m^{-2} (\sum_{i\leq m} \sqrt{\rho_i})^2.$$

On the other hand,

$$\begin{aligned} \sum_m m^{-2}(\sum_{i\leq m}\sqrt{\rho_i})^2 &\leq 2\sum_m m^{-2} \sum_{j\leq m}\sum_{i\leq j}\sqrt{\rho_i}\sqrt{\rho_j} \\ &= 2\sum_j \sqrt{\rho_j}(\sum_{i\leq j}\sqrt{\rho_i}) \sum_{m\geq j} m^{-2} \\ &\leq 4\sum_j j^{-1}\sqrt{\rho_j}(\sum_{i\leq j}\sqrt{\rho_i}) \\ &\leq 4(\sum_j \rho_j)^{1/2}(\sum_j (j^{-1}\sum_{i\leq j}\sqrt{\rho_i})^2)^{1/2}, \end{aligned}$$

so

$$\sum_m d_m \leq \sum_m m^{-2}(\sum_{i\leq m}\sqrt{\rho_i})^2 \leq 16\sum_j \rho_j.$$

Lemma 10 is proved.

By Lemma 10 and the assumption $\sum_n \rho_n < \pi/64$, we get $2\sum_m d_m < \pi/2$. So, we can choose $\{e_m\} \subset S^1$, such that $|e_{m+1} - e_m| = 2d_m$ and $0 < \arg e_m < \arg e_{m+1} < \pi/2$ for all m. In addition, by the fact $2d_m \geq \rho_m + \rho_{m+1}$, $\{I_m\}$ are disjoint mutually. We shall first apply induction to prove that

$$|e_m - e_n| \geq \frac{n(m-n)}{m}\rho_n \tag{27}$$

for $m > n$. For $m = n+1$, $|e_m - e_n| = 2d_n > \frac{n}{n+1}\rho_n = \frac{n(m-n)}{m}\rho_n$. Suppose $|e_m - e_n| \geq \frac{n(m-n)}{m}\rho_n$, we have

$$\begin{aligned} |e_{m+1} - e_n| &= |e_{m+1} - e_m| + |e_m - e_n| \\ &\geq 2d_m + \tfrac{n(m-n)}{m}\rho_n \geq \left(\tfrac{2n^2}{m^2} + \tfrac{n(m-n)}{m}\right)\rho_n \\ &> \tfrac{n(m+1-n)}{m+1}\rho_n. \end{aligned}$$

So, (27) holds for all $m > n$. Now, we shall prove that

$$\sup_{\theta \in S^1} \sum_{n:\theta \notin I_n} \frac{\rho_n^2}{|\theta - e_n|^2} < \infty. \tag{28}$$

For $\theta \in S^1$, we first consider

$$\begin{aligned} N_\theta^+ &\stackrel{def}{=} \{n : 0 \leq \arg\theta - \arg e_n \leq \pi/2\} \\ N_\theta^- &\stackrel{def}{=} \{n : 0 \geq \arg\theta - \arg e_n \geq -\pi/2\} \\ N_\theta^0 &\stackrel{def}{=} \{n : |\arg\theta - \arg e_n| > \pi/2\}. \end{aligned}$$

Label the elements in N_θ^+ by sub-index such that

$$\cdots < |\theta - e_{n_{-2}}| < |\theta - e_{n_{-1}}| < |\theta - e_{n_0}|$$

Then, $N_\theta^+ = \{\cdots < n_{-1} < n_0\}$ or $N_\theta^+ = \{n_{-K} < \cdots < n_{-1} < n_0\}$. By (27), in the second case, $|\theta - e_{n_{-l}}| > |e_{n_{-K}} - e_{n_{-l}}| > \frac{n_{-l}(n_{-K} - n_{-l})}{n_{-K}}\rho_{n_{-l}}$, thus

$$\begin{aligned} \sum_{n:n\in N_\theta^+, \theta\notin I_n} \tfrac{\rho_n^2}{|\theta - e_n|^2} &\leq 1 + \sum_{l=1}^{K-1} \left(\tfrac{n_{-l}(n_{-K} - n_{-l})}{n_{-K}}\right)^{-2} \\ &\leq \sum_{l=0}^{K-1} \left(\tfrac{n_l(n_K - n_l)}{n_K}\right)^{-2} \leq C < \infty; \end{aligned}$$

in the fist case, $|\theta - e_{n_{-l}}| > n_{-l}\rho_{n_{-l}}$, so

$$\sum_{n:n\in N_\theta^+, \theta\notin I_n} \frac{\rho_n^2}{|\theta - e_n|^2} \leq 1 + \sum_{l=1}^{\infty} (n_{-l})^{-2} = C < \infty. \tag{29}$$

Label the elements in N_θ^- by sub-index such that

$$|\theta - e_{n_0}| < |\theta - e_{n_1}| < |\theta - e_{n_2}| < \cdots$$

Then, $N_\theta^+ = \{n_0 < n_1 < \cdots\}$ or $N_\theta^+ = \{n_0 < n_1 < \cdots < n_K\}$. By (27), it is easy to show that

$$\sum_{n:n\in N_\theta^-,\theta\notin I_n} \frac{\rho_n^2}{|\theta - e_n|^2} \le C < \infty. \tag{30}$$

In addition,

$$\sum_{n:n\in N_\theta^0,\theta\notin I_n} \frac{\rho_n^2}{|\theta - e_n|^2} \le C \sum_{n:n\in N_\theta^0,\theta\notin I_n} \rho_n^2 = C' < \infty. \tag{31}$$

From (29)-(31), we get (28). By (28) and Theorem 2, we get Theorem 3.

Finally, we prove Corollary 4. By (7), we can choose $\{t_n\}$ such that $1 < t_1 < t_2 < \cdots$ and $\varphi(t_n) > 2^n t_n$. Set $\lambda_n = 2^{-n}$, $\rho_n = 2^{-n}t_n^{-1}$, $\Omega = \sum_n \lambda_n a_n$ where $\{a_n\}$ are H^1-atoms satisfying (4) and $|a_n(\theta)| = \rho_n^{-1}$ for $\theta \in I_n$. Then, $\Omega \in H^1(S^1)$, but

$$\int_{S^1} \varphi(|\Omega(\theta)|)d\theta = 2\sum_n \rho_n \varphi(\lambda_n \rho_n^{-1}) = \infty,$$

i.e. $\Omega \notin \varphi(L)$.

5 Appendix

In the proofs of Lemmas 8-9, we apply the estimates (15) and (19) without proofs. In what follows, we shall give details of their proofs along the ideas developped in [2], [3], [6], [8] and [9].

Proof of (15)

We have

$$\begin{aligned} \left\|\sum_j a_{n,j} * B_{n,j-s}\right\|_2^2 &= \sum_j \sum_i \langle a_{n,j} * B_{n,j-s}, a_{n,i} * B_{n,i-s}\rangle \\ &\le 2\sum_j \sum_{i\le j} |\langle B_{n,j-s}, \widetilde{a_{n,j}} * a_{n,i} * B_{n,i-s}\rangle| \\ &\le 2\sum_j \|B_{n,j-s}\|_1 \left\|\sum_{i\le j} |\widetilde{a_{n,j}} * a_{n,i} * B_{n,i-s}|\right\|_\infty \\ &\le 2\,\|f\|_1 \sup_j \left\|\sum_{i\le j} |\widetilde{a_{n,j}} * a_{n,i} * B_{n,i-s}|\right\|_\infty, \end{aligned} \tag{32}$$

where $\widetilde{a_{n,j}}(x)=a_{n,j}(-x)$. We first estimate $\sum_{i\le -3}|\widetilde{a_{n,0}}*a_{n,i}*B_{n,i-s}(0)|$. We have

$$\begin{aligned}\widetilde{a_{n,0}}*a_{n,i}*B_{n,i-s}(0) &= \int\limits_{R^2} B_{n,i-s}(y)(\int\limits_{R^2} a_{n,0}(y+z)a_{n,i}(z)dz)dy\\ &= \int\limits_{R^2} B_{n,i-s}(y)(\int\limits_{S^1}(\int\limits_0^{+\infty} a_n(\tfrac{y+t\theta}{|y+t\theta|})\tfrac{\beta(|y+t\theta|)}{|y+t\theta|^2}\tfrac{\beta(2^{-i}t)}{t}dt)a_n(\theta)d\theta)dy\\ &\overset{def}{=} \int\limits_{S^1} a_n(\theta)T_{i,s}(a_n)(\theta)d\theta\end{aligned}$$

where

$$\begin{aligned}T_{i,s}(a_n)(\theta) &= \int\limits_{R^2} B_{n,i-s}(y)L_i^y(a_n)(\theta)dy\\ L_i^y(a_n)(\theta) &= \int\limits_0^{+\infty} a_n(\tfrac{y+t\theta}{|y+t\theta|})\tfrac{\beta(|y+t\theta|)}{|y+t\theta|^2}\tfrac{\beta(2^{-i}t)}{t}dt.\end{aligned}$$

So, by (4),

$$\begin{aligned}\textstyle\sum_{i\le -3}|\widetilde{a_{n,0}}*a_{n,i}*B_{n,i-s}(0)| &\le \textstyle\sum_{i\le -3}|\int_{S^1} a_n(\theta)T_{i,s}(a_n)(\theta)d\theta|\\ &\le \textstyle\sup_\theta \sum_{i\le -3}|T_{i,s}(a_n)(\theta)|.\end{aligned} \tag{33}$$

For convenience, let

$$\begin{aligned}Q_n^*(\theta) &\overset{def}{=} \left\{y: \tfrac{1}{4}\le |y|\le 3, \left|\langle y,\theta^\perp\rangle\right|\le 4\rho_n\right\}\\ \Theta_{n,i,\theta} &\overset{def}{=} \left\{(y,t): a_n(\tfrac{y+t\theta}{|y+t\theta|})\tfrac{\beta(|y+t\theta|)}{|y+t\theta|^2}\tfrac{\beta(2^{-i}t)}{t}\ne 0\right\}\end{aligned} \tag{34}$$

where $\theta^\perp$ be one of the two unit vectors orthogonal to θ. Then

$$supp(L_i^y(a_n)(\theta))\subset Q_n^*(\theta). \tag{35}$$

Actually, if $L_i^y(a_n)(\theta)\ne 0$, $i\le -3$ and $(y,t)\in\Theta_{n,i,\theta}$, we have $|y|\le |y+t\theta|+|t\theta|\le 2+2^{i+1}\le 3$, $|y|\ge |y+t\theta|-|t\theta|\ge \frac{1}{2}-2^{i+1}\ge\frac{1}{4}$, and

$$\begin{aligned}\left|\langle y,\theta^\perp\rangle\right| &= \left|\langle y+t\theta,\theta^\perp\rangle\right|\le |y+t\theta|\left|\langle \tfrac{y+t\theta}{|y+t\theta|},\theta^\perp\rangle\right|\le |y+t\theta|\left|\tfrac{y+t\theta}{|y+t\theta|}-\theta\right|\\ &\le |y+t\theta|\left(\left|\tfrac{y+t\theta}{|y+t\theta|}-e_n\right|+|e_n-\theta|\right)\le 4\rho_n.\end{aligned}$$

Let

$$Q_s(\theta)=\left\{y: |y|\le 4 \text{ and } |\angle\langle y,\theta\rangle|\le \rho_n 2^{-\frac{s}{2}}\right\}$$

where $s>0$, then

$$\begin{aligned}\sum_{i\le -3}|T_{i,s}(a_n)(\theta)| &\le \sum_{i\le -3}|\textstyle\int_{R^2} B_{n,i-s}(y)L_i^y(a_n)(\theta)dy|\\ &\le \sum_{i\le -3}\Big(\sum_{j:Q_{n,j}\cap Q_s(\theta)\ne\emptyset,\overline{d}(Q_{n,j})=2^{i-s}}+\sum_{j:Q_{n,j}\cap Q_s(\theta)=\emptyset,\overline{d}(Q_{n,j})=2^{i-s}}\Big)\\ &\quad\cdot|\textstyle\int_{R^2} b_{n,j}(y)L_i^y(a_n)(\theta)dy|\overset{def}{=} I+II.\end{aligned} \tag{36}$$

For I, noticing that

$$|L_i^y(a_n)(\theta)| \le \int_0^{+\infty} \rho_n^{-1} \frac{\beta(|y+t\theta|)}{|y+t\theta|^2} \frac{\beta(2^{-i}t)}{t} dt \le C\rho_n^{-1},$$

we have

$$\begin{aligned} I &\le \sum_{i\le -3} \sum_{j:Q_{n,j}\cap Q_s(\theta)\neq\emptyset, \overline{d}(Q_{n,j})=2^{i-s}} C\rho_n^{-1} \tfrac{\lambda}{\lambda_n} |Q_{n,j}| \\ &\le C\rho_n^{-1} \tfrac{\lambda}{\lambda_n} \sum_{j:Q_{n,j}\cap Q_s(\theta)\neq\emptyset, \overline{d}(Q_{n,j})\le 2^{-s}} |Q_{n,j}| \\ &\le C\rho_n^{-1} \tfrac{\lambda}{\lambda_n} \rho_n 2^{-\frac{s}{2}} = C\tfrac{\lambda}{\lambda_n} 2^{-\frac{s}{2}}. \end{aligned} \tag{37}$$

To estimate II, for $y \in Q_{n,j} \cap Q_n^*(\theta)$ where $\overline{d}(Q_{n,j}) = 2^{i-s}$, $\exists y_\theta$ such that $y - y_\theta // \theta$, $y_\theta - y_Q // \theta^\perp$ where y_Q is the center of $Q_{n,j}$, thus

$$\begin{aligned} &|y - y_\theta| = |\langle y - y_Q, \theta\rangle| \le 2^{i-s} \\ &|y_\theta - y_Q| = \left|\langle y - y_Q, \theta^\perp\rangle\right| \le C2^{i-s}\rho_n \\ &\overline{yy_\theta} \cap Q_s(\theta) = \emptyset \\ &\overline{y_\theta y_Q} \cap Q_s(\theta) = \emptyset. \end{aligned} \tag{38}$$

By Lemma 11 below and (38), we have

$$\begin{aligned} &|L_i^y(a_n)(\theta) - L_i^{y_\theta}(a_n)(\theta)| \le 2^{i-s} \sup_{z\in\overline{yy_\theta}} \left|\tfrac{\partial}{\partial\theta} L_i^z(a_n)(\theta)\right| \le C2^{-s}\rho_n^{-1} \\ &\left|L_i^{y_\theta}(a_n)(\theta) - L_i^{y_Q}(a_n)(\theta)\right| \le 2^{i-s}\rho_n \sup_{z\in\overline{y_\theta y_Q}} \left|\tfrac{\partial}{\partial\theta^\perp} L_i^z(a_n)(\theta)\right| \le C2^{-\frac{s}{2}}\rho_n^{-1}. \end{aligned} \tag{39}$$

So,

$$\begin{aligned} II = &\sum_{i\le -3} \sum_{j:Q_{n,j}\cap Q_s(\theta)=\emptyset, \overline{d}(Q_{n,j})=2^{i-s}} \left|\textstyle\int_{R^2} b_{n,j}(y) L_i^y(a_n)(\theta) dy\right| \\ \le &\sum_{i\le -3} \sum_{j:Q_{n,j}\cap Q_s(\theta)=\emptyset, \overline{d}(Q_{n,j})=2^{i-s}} \textstyle\int_{R^2} |b_{n,j}(y)| \left|L_i^y(a_n)(\theta) - L_i^{y_Q}(a_n)(\theta)\right| dy \\ \le &\sum_{i\le -3} \sum_{j:Q_{n,j}\cap Q_s(\theta)=\emptyset, Q_{n,j}\cap Q_n^*(\theta)\neq\emptyset, \overline{d}(Q_{n,j})=2^{i-s}} C\tfrac{\lambda}{\lambda_n} |Q_{n,j}| \\ &\cdot \sup_y \left(|L_i^y(a_n)(\theta) - L_i^{y_\theta}(a_n)(\theta)| + \left|L_i^{y_\theta}(a_n)(\theta) - L_i^{y_Q}(a_n)(\theta)\right| \right) \\ \le &\; C2^{-\frac{s}{2}}\rho_n^{-1}\tfrac{\lambda}{\lambda_n} \sum_{j:Q_{n,j}\cap Q_n^*(\theta)\neq\emptyset, \overline{d}(Q_{n,j})\le 2^{-s}} |Q_{n,j}| \le C2^{-\frac{s}{2}}\tfrac{\lambda}{\lambda_n}. \end{aligned} \tag{40}$$

By (33), (36), (38) and (40), we get

$$\sum_{i\le -3} |\widetilde{a_{n,0}} * a_{n,i} * B_{n,i-s}(0)| \le C2^{-\frac{s}{2}} \frac{\lambda}{\lambda_n}.$$

By translation arguments,

$$\left\| \sum_{i\leq -3} |\widetilde{a_{n,0}} * a_{n,i} * B_{n,i-s}| \right\|_{L^\infty(R^2)} \leq C2^{-\frac{s}{2}} \frac{\lambda}{\lambda_n}. \tag{41}$$

By dilation arguments,

$$\left\| \sum_{i\leq j} |\widetilde{a_{n,j}} * a_{n,i} * B_{n,i-s}| \right\|_{\infty} \leq C2^{-\frac{s}{2}} \frac{\lambda}{\lambda_n}.$$

Combining with (32), it gives (15).

Lemma 11 *For $\theta \in supp(a_n)$, $\left|\langle y, \theta^\perp\rangle\right| \geq \rho_n 2^{-\frac{s}{2}}$ where $s > 0$, $|y| \leq 4$, we have*

$$\begin{array}{ll} (i) & \left|\frac{\partial}{\partial\theta} L_i^y(a_n)(\theta)\right| \leq 2^{-i}\rho_n^{-1} \\ (ii) & \left|\frac{\partial}{\partial\theta^\perp} L_i^y(a_n)(\theta)\right| \leq 2^{-i+\frac{s}{2}}\rho_n^{-2}. \end{array}$$

Proof. Without loss of generality, we may assume $\theta = (1,0)$. Let $e_1 = (1,0)$, $e_2 = (0,1)$, then $y = y_1 e_1 + y_2 e_2$, $|y_2| \geq \rho_n 2^{-\frac{s}{2}}$. Setting $w = \frac{y+t\theta}{|y+t\theta|} = \frac{y+te_1}{|y+te_1|}$, we have

$$J(w,y) = \left|\frac{dw}{dt}\right| = \frac{|y_2|}{|y+te_1|}. \tag{42}$$

Note that $\langle y + te_1, w^\perp\rangle = 0$, so $t = -\frac{\langle y, w^\perp\rangle}{\langle e_1, w^\perp\rangle}$. Obviously, for $(y,t) \in \Theta_{n,i,\theta}$ where $\Theta_{n,i,\theta}$ is defined by (34), we have

$$\begin{array}{l} \frac{\partial t}{\partial y_1} = -1 \\ \left|\frac{\partial t}{\partial y_2}\right| = \left|\frac{\langle e_2, w^\perp\rangle}{\langle e_1, w^\perp\rangle}\right| = \left|\frac{\langle e_1, w\rangle}{\langle e_2, w\rangle}\right| \leq C\rho_n^{-1} 2^{\frac{s}{2}}. \end{array} \tag{43}$$

We first estimate $\frac{\partial}{\partial y_1} L_i^y(a_n)(\theta)$. By (43), $\frac{\partial(y+te_1)}{\partial y_1} = 0 = \frac{\partial|y+te_1|}{\partial y_1}$. Thus,

$$\frac{\partial J(w,y)}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\frac{|y_2|}{|y+te_1|}\right) = 0,$$

and for $(y,t) \in \Theta_{n,i,\theta}$,

$$\left|\frac{\partial}{\partial y_1}\left(\frac{\beta(|y+te_1|)}{|y+te_1|^2}\frac{\beta(2^{-i}t)}{t}\right)\right| \leq C\left|\frac{\partial}{\partial y_1}\left(\frac{\beta(2^{-i}t)}{t}\right)\right| \leq C\frac{2^{-i}t+1}{t^2} \leq C2^{-2i}.$$

Therefore

$$\begin{array}{rcl} \left|\frac{\partial}{\partial y_1} L_i^y(a_n)(\theta)\right| & \leq & \int |a_n(w)| \left|\frac{\partial}{\partial y_1}\left(\frac{\beta(|y+t\theta|)}{|y+t\theta|^2}\frac{\beta(2^{-i}t)}{t}\right)\right| \frac{|dw|}{J(w,y)} \\ & & + \int |a_n(w)| \left|\frac{\beta(|y+t\theta|)}{|y+t\theta|^2}\frac{\beta(2^{-i}t)}{t}\right| \frac{\left|\frac{\partial J(w,y)}{\partial y_1}\right|}{J(w,y)} \frac{|dw|}{J(w,y)} \\ & \leq & C\rho_n^{-1} 2^{-2i} \int_{w([2^{i-1}, 2^{i+1}])} \frac{|dw|}{J(w,y)} \\ & \leq & C\rho_n^{-1} 2^{-2i} \int_{2^{i-1}}^{2^{i+1}} dt = C\rho_n^{-1} 2^{-i}. \end{array} \tag{44}$$

Now, we estimate $\left|\frac{\partial}{\partial y_2} L_i^y(a_n)(\theta)\right|$. At first, by (43), we have

$$\begin{aligned} \left|\frac{\partial|y+te_1|}{\partial y_2}\right| &= \frac{\left|\langle y+te_1, \frac{\partial}{\partial y_2}(y+te_1)\rangle\right|}{|y+te_1|} \le \left|\frac{\partial}{\partial y_2}(y+te_1)\right| \\ &\le C(\left|\frac{\partial t}{\partial y_2}\right|+1) \le C\rho_n^{-1}2^{\frac{s}{2}}. \end{aligned} \tag{45}$$

Thus, by (43) and (45), for $(y,t) \in \Theta_{n,i,\theta}$,

$$\begin{aligned} &\left|\frac{\partial}{\partial y_2}(\frac{\beta(2^{-i}t)}{t})\right| \le C\frac{2^{-i}t+1}{t^2}\left|\frac{\partial t}{\partial y_2}\right|\chi_{2^{i-1}\le t\le 2^{i+1}}(t) \le C2^{-2i}\rho_n^{-1}2^{\frac{s}{2}} \\ &\left|\frac{\partial}{\partial y_2}(\frac{\beta(|y+te_1|)}{|y+te_1|^2})\right| \le C\frac{|y+te_1|^2+|y+te_1|}{|y+te_1|^4}\left|\frac{\partial(|y+te_1|)}{\partial y_2}\right|\chi_{\frac{1}{2}\le|y+te_1|\le 2} \le C\rho_n^{-1}2^{\frac{s}{2}}, \end{aligned}$$

which means that

$$\left|\frac{\partial}{\partial y_2}(\frac{\beta(|y+te_1|)}{|y+te_1|^2}\frac{\beta(2^{-i}t)}{t})\right| \le C2^{-2i}\rho_n^{-1}2^{\frac{s}{2}} \tag{46}$$

In addition, for $(y,t) \in \Theta_{n,i,\theta}$ such that $\left|\langle y, \theta^{\perp}\rangle\right| \ge \rho_n 2^{-\frac{s}{2}}$, we have

$$\begin{aligned} \left|\frac{\partial J(w,y)}{\partial y_2}\right| &= \left|\frac{\partial}{\partial y_2}(\frac{|y_2|}{|y+te_1|})\right| \le \frac{\left|\frac{\partial(|y_2|)}{\partial y_2}\right||y+te_1|+|y_2|\left|\frac{\partial(|y+te_1|)}{\partial y_2}\right|}{|y+te_1|^2} \\ &\le C(1+|y_2|\rho_n^{-1}) \end{aligned}$$

thus,

$$\left|\frac{\partial J(w,y)}{\partial y_2}J(w,y)\right| \le \frac{C(1+|y_2|\rho_n^{-1})}{|y_2|} \le C\rho_n^{-1}2^{\frac{s}{2}}. \tag{47}$$

By (46) and (47), we get

$$\begin{aligned} \left|\frac{\partial}{\partial y_2}L_i^y(a_n)(e_1)\right| \le\ & \int |a_n(w)|\left|\frac{\partial}{\partial y_2}(\frac{\beta(|y+t\theta|)}{|y+t\theta|^2}\frac{\beta(2^{-i}t)}{t})\right|\frac{|dw|}{J(w,y)} \\ & +\int |a_n(w)|\left|\frac{\beta(|y+t\theta|)}{|y+t\theta|^2}\frac{\beta(2^{-i}t)}{t}\right|\frac{\left|\frac{\partial J(w,y)}{\partial y_2}\right|}{J(w,y)}\frac{|dw|}{J(w,y)} \\ \le\ & C2^{-2i+\frac{s}{2}}\rho_n^{-1}\rho_n^{-1}\int_{w([2^{i-1},2^{i+1}])}\frac{|dw|}{J(w,y)} \\ & +C\rho_n^{-1}2^{\frac{s}{2}}\rho_n^{-1}\int_{w([2^{i-1},2^{i+1}])}\frac{|dw|}{J(w,y)} \\ \le\ & C2^{-2i+\frac{s}{2}}\rho_n^{-2}\int_{2^{i-1}}^{2^{i+1}}dt = C2^{-i+\frac{s}{2}}\rho_n^{-2}. \end{aligned}$$

Lemma 11 is proved.

Proof of (19)

For simplicity, we omit the subindex n. By the definition of φ^s (i.e. φ_n^s , see (13)) and ψ, we have

$$(\delta-\varphi^s)^\wedge(\cdot) = 1-\psi(2^{-\frac{s}{6}}\rho^{-1}\langle\frac{\cdot}{|\cdot|},e\rangle) = \sum_{m\ge 0}\phi(2^{-\frac{s}{6}-m}\rho^{-1}\langle\frac{\cdot}{|\cdot|},e\rangle)$$

where $\phi \in C_0^\infty((-2,-\frac{1}{2}) \cup (\frac{1}{2},2))$ is defined by

$$\phi|_{\left\{u:\frac{1}{2}<|u|\leq 1\right\}} = 1-\psi,\ \phi|_{\{u:1\leq|u|<2\}} = \psi(\frac{\cdot}{2}),\ \phi|_{\left\{u:\frac{1}{2}<|u|<2\right\}^c} = 0$$

which satisfies $\sum_{m\geq 0}\phi(2^{-m}u) = 1-\psi(u)$ for all $u \in R^1$. Take a nonnegative function $\varsigma \in C_0^\infty(\left\{\frac{1}{2} < |\cdot| < 2\right\})$ such that $\sum_{-\infty}^{+\infty}\varsigma^2(2^{-k}u) = 1$ for all $u \in R^1$, and set $L_k = (\varsigma(2^{-k}\cdot))^\vee$, then

$$\begin{aligned} a_j * (\delta - \varphi^s) &= \sum_{m\geq 0}\sum_k (C_m^{s,k})^\vee * a_j * L_k \\ C_m^{s,k}(y) &= \varsigma(2^{-k}y)\phi(2^{-\frac{s}{6}-m}\rho^{-1}\langle \tfrac{y}{|y|}, e\rangle) \end{aligned} \tag{48}$$

where a_j is just $a_{n,j}$ defined by (13) with center $e = e_n$ and radius $\rho = \rho_n$.

We first estimate $\left\|(C_m^{s,k})^\vee\right\|_1$. Define an inversible linear operator $A_{k.m} : R^2 \to R^2$ by

$$A_{k,m}y = (2^{k+\frac{s}{6}+m}\rho y_1, 2^k y_2) \text{ where } y_1//e \text{ and } y_2 \perp e.$$

Let $h = 2^{\frac{s}{6}+m}\rho$, by Sobolev imbedding theorem ($\left\|\widehat{f}\right\|_1 \leq C\sum_{|\alpha|\leq 2}\|\partial^\alpha f(Q(\cdot))\|_2$ where Q is a nonsingular linear transform on R^2)

$$\begin{aligned} \left\|(C_m^{s,k})^\vee\right\|_1 &\leq C\sum_{|\alpha|\leq 2}\left\|\partial_y^\alpha\left(C_m^{s,k}(A_{k,m}y)\right)\right\|_2 \\ &= C\sum_{|\alpha|\leq 2}\left\|\partial_y^\alpha\left(\varsigma(hy_1+y_2)\phi(\tfrac{\langle y_1,e\rangle}{|hy_1+y_2|})\right)\right\|_2. \end{aligned} \tag{49}$$

In the supports of ς and ϕ, we have

$$\frac{1}{2} \leq |hy_1+y_2| \leq 2, \frac{1}{2} \leq \frac{|y_1|}{|hy_1+y_2|} \leq 2,$$

which means that

$$|y_1| \leq 4, |y_2| \leq 2, h \leq \frac{|hy_1+y_2|}{|y_1|} \leq 2, \tag{50}$$

so

$$\left|\partial_y^\alpha\left(\varsigma(hy_1+y_2)\phi(\frac{\langle y_1,e\rangle}{|hy_1+y_2|})\right)\right| \leq C. \tag{51}$$

From (49)-(51), we get

$$\left\|(C_m^{s,k})^\vee\right\|_1 \leq C\sum_{|\alpha|\leq 2}\left\|\partial_y^\alpha\left(\varsigma(hy_1+y_2)\phi(\frac{\langle y_1,e\rangle}{|hy_1+y_2|})\right)\right\|_2 \leq C. \tag{52}$$

Now,

$$\begin{aligned}
\|a_j * L_k\|_1 &= \int \left| \int \frac{a(y/|y|)\beta(2^{-j}|y|)}{|y|^2} L_k(x-y)dy \right| dx \\
&= \int \left| \int_{S^1} \int_0^\infty \beta(2^{-j}r) \frac{a(\theta)}{r} L_k(x-r\theta) dr d\theta \right| dx \\
&\le \int_{R^2} \int_{S^1} \int_{2^{j-1}}^{2^{j+1}} 2^{-j} |a(\theta)| \, |L_k(x-r\theta) - L_k(x-re)| \, dr d\theta dx \\
&\le C \sup_{r\in[2^{j-1},2^{j+1}],\theta\in supp(a)} \int_{R^2} |L_k(x-r\theta) - L_k(x-re)| \, dx \\
&\le C \sup_{r\in[2^{j-1},2^{j+1}],\theta\in supp(a)} r\,|\theta - e| \, \|\nabla L_k\|_1 \le C2^{k+j}\rho.
\end{aligned} \tag{53}$$

By (52)-(53),

$$\left\| (C_m^{s,k})^\vee * a_j * L_k \right\|_1 \le C2^{k+j}\rho. \tag{54}$$

But, for larger k (say, $k+j \ge 0$), the estimate (54) is not enough, we need some other estimate.

Applying Sobolev imbedding thorem again, we get

$$\begin{aligned}
\left\| (C_m^{s,k})^\vee * a_j * L_k \right\|_1 &\le C \sum_{|\alpha|\le 2} \left\| \partial_y^\alpha \left(C_m^{s,k}(A_{k,m}y) \widehat{a_j}(A_{k,m}y) \right) \right\|_2 \\
&\le C \sum_{|\alpha|\le 2} \left\| \partial_y^\alpha \left(\widehat{a_j}(A_{k,m}\cdot) \right) \chi_{supp(C_m^{s,k}(A_{k,m}\cdot))} \right\|_2
\end{aligned} \tag{55}$$

by (50)-(51). Write

$$\begin{aligned}
& y = y_1 e + y_2 e^\perp \in supp(C_m^{s,k}(A_{k,m}\cdot)), \\
& \theta = \theta_1 e + \theta_2 e^\perp \in supp(a), |\theta_1| \le 1, |\theta_2| \le \rho.
\end{aligned} \tag{56}$$

Note that

$$\begin{aligned}
& \varsigma(2^{-k}A_{k,m}y) \ne 0 \Longrightarrow |A_{k,m}y| \sim 2^k \\
& \phi(2^{-\frac{s}{6}-m}\rho^{-1}\langle \tfrac{A_{k,m}y}{|A_{k,m}y|}, e\rangle) \ne 0 \Longrightarrow \left|\langle \tfrac{A_{k,m}y}{|A_{k,m}y|}, e\rangle\right| \sim 2^{\frac{s}{6}+m}\rho.
\end{aligned}$$

In addition, $|\theta - e| < \rho$ and $\frac{s}{6}+m > 0$, so

$$\begin{aligned}
& \left|\langle \tfrac{A_{k,m}y}{|A_{k,m}y|}, \theta\rangle\right| \sim 2^{\frac{s}{6}+m}\rho \\
& |\langle A_{k,m}y, \theta\rangle| \sim 2^{\frac{s}{6}+m+k}\rho,
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
\left| \partial_y^\alpha \langle A_{k,m}y, \theta\rangle \right| &= \left| \partial_y^\alpha \left(\theta_1 2^{\frac{s}{6}+m+k}\rho y_1 + \theta_2 2^k y_2 \right) \right| \\
&\le C(2^{\frac{s}{6}+m+k}\rho)^{\alpha_1} \cdot (2^k\rho)^{\alpha_2} \le C(2^{\frac{s}{6}+m+k}\rho)^{|\alpha|}.
\end{aligned} \tag{58}$$

Now, by integration by parts,

$$\begin{aligned}\widehat{a_j}(A_{k,m}y) &= \int a(\theta)\int \frac{\beta(2^{-j}r)}{r}e^{-ir\langle A_{k,m}y,\theta\rangle}drd\theta \\ &= \int \frac{a(\theta)}{(-i\langle A_{k,m}y,\theta\rangle)^3}\int \partial_r^3(\frac{\beta(2^{-j}r)}{r})e^{-ir\langle A_{k,m}y,\theta\rangle}drd\theta.\end{aligned}$$

So, by (57)-(58) and the fact $\left|\partial_r^3(\frac{\beta(2^{-j}r)}{r})r^{|\alpha|-1}\right| \leq C2^{-j(3-|\alpha|+1)}$, we have

$$\begin{aligned}\left|\partial_y^\alpha\left(\widehat{a_j}(A_{k,m}y)\right)\right| &\leq C\sum_{0\leq\gamma\leq\alpha}\left|\int \frac{a(\theta)(-ir)^{|\gamma|}\partial_y^\gamma\langle A_{k,m}y,\theta\rangle}{(-i\langle A_{k,m}y,\theta\rangle)^3}\int \partial_r^3(\frac{\beta(2^{-j}r)}{r})e^{-ir\langle A_{k,m}y,\theta\rangle}drd\theta\right| \\ &\leq C\sum_{0\leq\gamma\leq\alpha}\rho\rho^{-1}(2^{\frac{s}{6}+m+k}\rho)^{-3}2^{-j(3-|\gamma|+1)}2^j(2^{\frac{s}{6}+m+k}\rho)^{|\gamma|} \\ &= C\sum_{0\leq\gamma\leq\alpha}(2^{\frac{s}{6}+m+k+j}\rho)^{-3+|\gamma|}.\end{aligned} \tag{59}$$

By (55), (50) and (59), we have

$$\left\|(C_m^{s,k})^\vee * a_j * L_k\right\|_1 \leq C\sum_{|\alpha|\leq 2}(2^{\frac{s}{6}+m+k+j}\rho)^{-3+|\alpha|} = C\sum_{l=1}^{3}(2^{\frac{s}{6}+m+k+j}\rho)^{-l}. \tag{60}$$

So,

$$\begin{aligned}\left\|\sum_{m\geq 0}\sum_k(C_m^{s,k})^\vee * a_j * L_k\right\|_1 &\leq C\sum_{m\geq 0}\sum_k\min\left\{2^{k+j}\rho,\sum_{l=1}^{3}(2^{\frac{s}{6}+m+k+j}\rho)^{-l}\right\} \\ &\leq C\sum_{m\geq 0}\sum_{k:2^{k+j}\rho\leq 2^{-\frac{s}{12}-\frac{m}{2}}}2^{k+j}\rho \\ &\quad +C\sum_{m\geq 0}\sum_{k:2^{k+j}\rho>2^{-\frac{s}{12}-\frac{m}{2}}}\sum_{l=1}^{3}(2^{\frac{s}{6}+m+k+j}\rho)^{-l} \\ &\leq C\sum_{m\geq 0}2^{-\frac{s}{12}-\frac{m}{2}} = C2^{-\frac{s}{12}}.\end{aligned} \tag{61}$$

From (48), (61), we get (19).

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