L^2-boundedness of Hilbert transforms along variable curves*

Jiecheng CHEN & Xiangrong ZHU
Dept of Mathematics (Xixi Campus), Zhejiang Universitry

Abstract

For the L^2-boundedness of the Hilbert transforms along variable curves

$$H_{\phi,\gamma}(f)(x_1,x_2)=p.v.\int_{-\infty}^{+\infty}f(x_1-t,x_2-\phi(x_1)\gamma(t))\frac{dt}{t}$$

where $\gamma\in C^2(R^1)$, odd or even, $\gamma(0)=\gamma'(0)=0$, convex on $(0,\infty)$, if $\phi\equiv 1$, A. Nagel, J. Vance, S. Wainger and D. Weinberg got a necessary and sufficient condition on γ; if ϕ is a polynomial, J. M. Bennett got a sufficient condition on γ. In this paper, we shall first give a counter-example to show that under the condition of Nagel-Vance-Wainger-Weinberg on γ, the L^2-boundedness of $H_{\phi,\gamma}$ may fail even if $\phi\in C^\infty(R^1)$. Then, we improve Bennett's result by relaxing the condition on γ and simplifying the proof.

*Supported by 973 project and NSFZJ

1 Introduction

For $f \in \mathcal{S}(R^2)$ and $\Phi : R^3 \to R^2$, define a Hilbert transform along variable curves by

$$H_\Phi(f)(x) = p.v. \int_{-\infty}^{+\infty} f(x - \Phi(x,t)) \frac{dt}{t}. \tag{1}$$

It is well known that such kind of operators have been studied extensively. For example, see [7] and [2] and [12] for the case $\Phi(x,t) = (t, \gamma(t))$, [10] and [4] for the case $\Phi(x,t) = (t, S(x_1, x_1 - t))$, [9] for the case $\Phi(x,t) = (t, P(x_1, t))$ where P is a polynomial on R^2, [5] for the case $\Phi(x,t) = (t, x_1 t)$, [3] for the case $\Phi(x,t) = tv(x)$ where $v : R^2 \to R^2$ is a vector field, [1] for the case $\Phi(x,t) = (t, P(x_1)\gamma(t))$ where P is a real polynomial on R^1.

In this paper, we shall consider its L^2-boundedness. Some of related known results are the following.

Theorem 1 *For $\Phi(x,t) = (t, \gamma(t))$ where*

$$\gamma \in C^2(R^1), \gamma(0) = 0, \textit{ convex on } (0, \infty), \tag{2}$$

H_Φ is L^2-bounded iff

$$\begin{array}{ll} (a) & \textit{when } \gamma \textit{ is odd, } h(ct) \geq 2h(t) (\forall t \in (0,\infty) \textit{ for some } c > 1, \\ & \textit{where } h(t) = t\gamma'(t) - \gamma(t) \textit{ , or} \\ (b) & \textit{when } \gamma \textit{ is even, } \gamma'(ct) \geq 2\gamma'(t) \ (\forall t \in (0,\infty) \textit{ for some } c > 1. \end{array} \tag{3}$$

See [7]. And note that (2)-(3) imply $\gamma'(0) = 0$, and the condition "$\gamma'(ct) \geq 2\gamma'(t)$" implies the condition "$h(ct) \geq 2h(t)$".

Theorem 2 *For $\Phi(x,t) = (t, \phi(x_1)\gamma(t))$ where $\phi = P$ is a real polynomial,*

$$\gamma \in C^3(R^1) \textit{ is odd or even, convex on } (0,\infty), \textit{ and } \gamma(0) = \gamma'(0) = 0, \tag{4}$$

$$\lambda(t) = t(\gamma''(t)/\gamma'(t)) \textit{ is decreasing and positively bounded below on } (0,\infty), \tag{5}$$

then, H_Φ is L^2-bounded.

See [1]. Also see [5] for $\phi(x_1) = x_1$.

In this paper, we shall prove the following theorems.

Theorem 3 *For $\Phi(x,t) = (t, \phi(x_1)\gamma(t))$, there are $\phi \in C^\infty(R^1)$ and γ (odd or even) satisfying (2)-(3) such that $H_{\phi,\gamma}$ ($\equiv H_\Phi$) is not $L^2(R^2)-$bounded.*

Theorem 4 *For $\Phi(x,t) = (t, \phi(x_1)\gamma(t))$, if $\phi = P$ is a real polynomial on R^1, $\gamma \in C^2(R^1)$, and*

$$\gamma(0) = \gamma'(0) = 0,\ \gamma''(t) > 0 \text{ for } t \in (0,\infty), \gamma \text{ is odd or even,} \tag{6}$$

and there are positive numbers λ and M such that

$$\left|\frac{\gamma''(s)}{\gamma'(s)} - \frac{\gamma''(t)}{\gamma'(t)}\right| \geq \frac{\lambda(s-t)^M}{(s+t)^{M+1}} \text{ for } 0 < t < s, \tag{7}$$

then, $\|H_{P,\gamma}(f)\|_2 \leq C\,\|f\|_2$ where C depends only on λ, M and the degree of P.

Theorem 3 shows that in Theorem 1, if $\gamma(t)$ is replaced by $\phi(x_1)\gamma(t)$, the result shall fail even if ϕ is a real analytic function. And Theorem 4 gives a weaker condition than Theorem 2, and our proof (by dealing with $R_\mu \circ R_\mu^*$ (see (45))) shall be much simpler than the proof (by dealing with $R_\mu^* \circ R_\mu$) given in [1]. Note that

- (5)=>(7) because (5) implies that for $0 < t < s$,

$$\frac{\gamma''(s)}{\gamma'(s)} - \frac{\gamma''(t)}{\gamma'(t)} \geq \frac{s-t}{st} \cdot \frac{s\gamma''(s)}{\gamma'(s)} \geq \frac{4\lambda_0(s-t)}{(s+t)^2}$$

  where $\lambda_0 = \inf_{t>0} t(\gamma''(t)/\gamma'(t))$.

- (7)=>(3) because (7) implies that $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing and for $t > 0$,

$$\frac{\gamma''(t)}{\gamma'(t)} \geq \frac{\gamma''(2t)}{\gamma'(2t)} + \frac{\lambda}{3^M} \cdot \frac{1}{t} \geq \cdots \geq \frac{\gamma''(2^k t)}{\gamma'(2^k t)} + \frac{\lambda}{3^M} \cdot \frac{1}{t} \cdot \sum_{j=0}^{k-1} 2^{-j} \geq \frac{2\lambda \cdot 3^{-M}}{t}$$

  which means that $\gamma'(2t) \geq (1 + \lambda \cdot 3^{-M})\gamma'(t)$, so (3) holds.

- (7)$\not=>$(5) for $\gamma(t) = t^3 e^{|t|}$.

- for $\gamma \in C^3(R^1) \cap (6)$, if

$$(\gamma''(t)/\gamma'(t))' \leq -\lambda/t^2 \tag{8}$$

  for all $t \geq 0$ and some $\lambda > 0$, then $\gamma \in (7)$. And, if $\gamma \in (5)$, then $\gamma \in (8)$.

2 Some Lemmas

Before proving the above theorems 3-4, we first give some lemmas.

Lemma 5 *Suppose S is a linear bounded operator from $L^p(R^d)$ to itself, $K \in \mathcal{S}(R^{2d})$ is its kernel which satisfies that $K \in L_{loc}(R^{2d} - \left\{(x,x) : x \in R^d\right\})$ and*

$$\sup_{y \in R^n} \int_{1 \le |x-y| \le 2} |K(x,y)|\, dx \le C, \tag{9}$$

then, the operator

$$S^1(f)(x) = \int_{|y| \le 1} K(x,y) f(y) dy$$

is also $L^p(R^d)$-bounded and $\|S^1\|_{p,p} \le C'(1 + \|S\|_{p,p})$.

Proof Take $\varphi \in C_0^\infty(R^d)$ which satisfies that $\varphi(y) = 1$ for $|y| \le 1$, $\varphi(y) = 0$ for $|y| \ge 2$, and $0 \le \varphi \le 1$. Define $S_\varphi : \mathcal{S}(R^d) \to \mathcal{S}'(R^d)$ by$< S_\varphi(f), g > = < K(\cdot, \circ)\varphi(\cdot - \circ), g(\cdot) f(\circ) >$ for all f and $g \in \mathcal{S}(R^d)$. By Lemma 7 in [1], we have that $\|S_\varphi\|_{p,p} \le \|\hat{\varphi}\|_1 \|S\|_{p,p}$. On the other hand, we have

$$(S^1 - S_\varphi)(f)(x) \le \int_{1 \le |x-y| \le 2} |K(x,y)|\, |f(y)|\, dy$$

which implies that $\|S^1 - S_\varphi\|_{p,p} \le C$ by (9). Thus, $\|S^1\|_{p,p} \le C + \|\hat{\varphi}\|_1 \|S\|_{p,p} \le C'(1 + \|S\|_{p,p})$. Lemma 5 is proved.

Now, let

$$\widetilde{f}(x,\lambda) = \textstyle\int_{R^1} f(x,s) e^{-i\lambda s} ds$$
$$\widetilde{T}(f)(x,\lambda) = p.v. \textstyle\int_{R^1} \widetilde{f}(x-y,\lambda) e^{-i\lambda\phi(x)\gamma(y)} \frac{dy}{y}.$$

By Fourier transform and Plancherel's formula (see [8] p116), we have

$$\|H_{\phi,\gamma}(f)\|^2_{L^2(R^2)} = \int_{R^1} \left\|\widetilde{T}(\widetilde{f})(\cdot,\lambda)\right\|^2_{L^2(R^1)} d\lambda. \tag{10}$$

So, if $H_{\phi,\gamma}$ is $L^2(R^2)-$bounded, $\widetilde{T}$ is also $L^2(R^2)-$bounded. In addition, the $L^2(R^2)-$boundedness of $\widetilde{T}$ means the $L^2(R^1)-$boundedness of T_λ for almost all $\lambda \in R^1$, where

$$T_\lambda(g)(x) = p.v. \int_{R^1} g(x-y) e^{-i\lambda\phi(x)\gamma(y)} \frac{dy}{y}.$$

Lemma 6 *For $\theta > 0, 0 < t_1 < t_2$, we have*

$$\frac{\gamma'(t_2)}{\gamma'(\theta + t_2)} - \frac{\gamma'(t_1)}{\gamma'(\theta + t_1)} \ge c_{\lambda,M} \frac{\gamma'(t_2)}{\gamma'(\theta + t_2)} \cdot \frac{\theta^M (t_2 - t_1)}{(\theta + t_2)^{M+1}} \tag{11}$$

where $c_{\lambda,M}$ depends only on λ and M.

Proof If $\frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} \leq \frac{1}{2}\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)}$, noticing that $\frac{\theta^M(t_2-t_1)}{(\theta+t_2)^{M+1}} \leq 1$, we have

$$\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)} - \frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} \geq \frac{1}{2}\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)} \geq \frac{1}{2}\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)} \cdot \frac{\theta^M(t_2-t_1)}{(\theta+t_2)^{M+1}}.$$

If $\frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} \geq \frac{1}{2}\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)}$, noticing that

$$\begin{aligned} \frac{\partial}{\partial t}(\frac{\gamma'(t)}{\gamma'(\theta+t)}) &= \frac{\gamma"(t)\gamma'(\theta+t)-\gamma\prime(t)\gamma"(\theta+t)}{(\gamma'(\theta+t))^2} \\ &= \frac{\gamma'(t)}{\gamma'(\theta+t)}(\frac{\gamma"(t)}{\gamma'(t)} - \frac{\gamma"(\theta+t)}{\gamma'(\theta+t)}) \geq \frac{\lambda\theta^M}{(\theta+2t)^{M+1}}\frac{\gamma'(t)}{\gamma'(\theta+t)} > 0 \end{aligned}$$

and for $t \in (t_1, t_2)$,

$$\frac{\partial}{\partial t}(\frac{\gamma'(t)}{\gamma'(\theta+t)}) \geq \frac{\lambda\theta^M}{(\theta+2t_2)^{M+1}}\frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} \geq \frac{\lambda}{2^{M+1}}\frac{\theta^M}{(\theta+t_2)^{M+1}}\frac{\gamma'(t_1)}{\gamma'(\theta+t_1)},$$

we get

$$\begin{aligned} \frac{\gamma'(t_2)}{\gamma'(\theta+t_2)} - \frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} &\geq \frac{\lambda}{2^{M+1}}\frac{(t_2-t_1)\theta^M}{(\theta+t_2)^{M+1}}\frac{\gamma'(t_1)}{\gamma'(\theta+t_1)} \\ &\geq \frac{\lambda}{2^{M+2}}\frac{(t_2-t_1)\theta^M}{(\theta+t_2)^{M+1}}\frac{\gamma'(t_2)}{\gamma'(\theta+t_2)}. \end{aligned}$$

Lemma 6 is proved.

Lemma 7 *For $n > 3$, there is a $g \in C^2([0,n])$ such that $0 < g'''(t) \in L^\infty([0,n])$ for $t \in (0,n)$, and*

$$\begin{aligned} &(a) \quad g(0) = g'(0) = g''(0) = 0, \\ &(b) \quad g(n) = g'(n) = g''(n) = 1. \end{aligned} \tag{12}$$

Proof We only need to choose

$$g(t) = 1 + (t-n) + \frac{1}{2}(t-n)^2 + \frac{1}{2}\int_n^t h(s)(t-s)^2 ds$$

and a suitable h such that

$$\begin{aligned} &(a) \quad 0 < h(t) \in ([0,n]) \text{ for } t \in (0,n) \\ &(b) \quad \textstyle\int_0^n h(s)ds = 1 \\ &(c) \quad \textstyle\int_0^n h(s)sds = n-1 \\ &(d) \quad \textstyle\int_0^n h(s)s^2 ds = n^2 - 2n + 2. \end{aligned} \tag{13}$$

Now, take

$$h(s) = \begin{cases} \mu & \text{for} \quad 0 < s < \epsilon \\ \nu & \text{for} \quad \epsilon < s < n \end{cases} \tag{14}$$

where ϵ, μ, ν are to be determined. From (13)-(14), we get

$$\begin{array}{ll} (a) & 1 = \epsilon(\mu - \nu) + n\nu \\ (b) & n - 1 = \frac{\epsilon^2}{2}(\mu^2 - \nu^2) + \frac{1}{2}n^2\nu \\ (b) & n^2 - 2n + 2 = \frac{\epsilon^3}{3}(\mu^3 - \nu^3) + \frac{1}{3}n^3\nu. \end{array} \tag{15}$$

$(15)(b) - \frac{n}{2}(15)(a)$ and $(15)(c) - \frac{2n}{3}(15)(b)$ imply

$$\begin{array}{l} n - 2 = \epsilon(\mu - \nu)(\epsilon - n), \\ n^2 - 4n + 6 = \epsilon^2(\mu - \nu)(\epsilon - n) \end{array} \tag{16}$$

which means

$$\epsilon = \frac{n^2 - 4n + 6}{n - 2} > 0. \tag{17}$$

From (16) and (17), we get

$$\mu - \nu = \frac{n - 2}{\epsilon(\epsilon - n)} = -\frac{(n - 2)^3}{2(n - 3)(n^2 - 4n + 6)}.$$

So,

$$\begin{array}{l} \nu = \frac{1}{n}(1 - \epsilon(\mu - \nu)) = \frac{n^2 - 2n + 2}{2n(n-3)} > 0 \\ \mu = \frac{1}{\epsilon}(1 - \nu(n - \epsilon)) = \frac{2}{n(n^2 - 4n + 6)} > 0. \end{array} \tag{18}$$

For the ϵ, μ, ν determined in (17)-(18) and (14), we get the desired h.

Lemma 8 *(Van der Corput's Lemma) If ϕ' is monotone on (a, b) and $\phi' \geq \lambda > 0$, then*

$$\left| \int_a^b e^{i\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1} \left(\|\psi\|_\infty + \int_a^b |\psi'(x)| \, dx \right). \tag{19}$$

See [11], p344.

3 Proof of the Theorems

3.1 Proof of Theorem 3

Take

$$\phi(x) = e^x$$
$$\gamma(y) = \begin{cases} e^n g_n(y) & \text{for} \quad 0 \leq y \leq n \\ e^y & \text{for} \quad y > n \\ \gamma(-y) & \text{for} \quad y < 0 \end{cases}$$

where n is a large positive number, and g_n is determined by Lemma 7. Then $\phi \in C^\infty(R^1)$, $\gamma \in C^2(R^1)$, $\gamma(0) = \gamma'(0) = 0$, $\gamma''(t) > 0$ for $t > 0$ and γ satisfies the condition (3)(b) (note that γ satisfies the condition (3)(a) if we choose $\gamma(y) = -\gamma(-y)$ for $y < 0$). Now we shall prove that for the above selected ϕ and γ, $H_{\phi,\gamma}$ is not $L^2(R^2)-$ bounded. Otherwise, almost all T_λ are $L^2(R^1)-$bounded, say, T_1 is $L^2(R^1)-$bounded. For T_1, we have

$$\begin{aligned} T_1(f)(x) &= p.v. \textstyle\int_{-\infty}^{+\infty} e^{-i\phi(x)\gamma(y)} f(x-y)\frac{dy}{y} \\ &= \left(p.v. \textstyle\int_{-n}^{n} + \int_{|y|\geq n}\right) e^{-i\phi(x)\gamma(y)} f(x-y)\frac{dy}{y} \\ &= S(f)(x) + R(f)(x) \end{aligned} \tag{20}$$

By Lemma 5, S is $L^2(R^1)-$bounded. So, R is $L^2(R^1)-$ bounded, and thus $L = R \circ R^*$ is also $L^2(R^1)-$bounded. Now,

$$\begin{aligned} &L(f)(x) = \textstyle\int_{R^1} L(x,y) f(y) dy \\ &L(x,y) = \textstyle\int_{|x-z|>n, |y-z|>n} e^{-i(\phi(x)\gamma(x-z) - \phi(y)\gamma(y-z))} \frac{dz}{(x-z)(y-z)}. \end{aligned} \tag{21}$$

For $y > 0$ and $x < -2n$, we have

$$\begin{aligned} L(x,y) = \; & \left(\textstyle\int_{x-z>n} + \int_{x-z<-n, y-z>n} + \int_{y-z<-n}\right) \cdot \\ & e^{-i(\phi(x)\gamma(x-z) - \phi(y)\gamma(y-z))} \frac{dz}{(x-z)(y-z)} \\ = \; & I + II + III \end{aligned} \tag{22}$$

If $y > 0$, $x < -2n$, $x - z > n$, then $(e^{2x-z} - e^{2y-z})''_{zz} < 0$ and $(e^{2x-z} - e^{2y-z})'_z \geq e^y$. So, by Van der Corput's Lemma, we have

$$|I| = \left| \int_{x-z>n} e^{-i(e^{2x-z} - e^{2y-z})} \frac{dz}{(x-z)(y-z)} \right| \leq Ce^{-y}. \tag{23}$$

If $y > 0$, $x < -2n$, $x - z < -n$ and $y - z > n$, then $(e^z - e^{2y-z})'_z \geq e^y$ and $(e^z - e^{2y-z})_z$ is increasing for $z > y$ and decreasing for $z < y$. So, by Van der Corput's Lemma, we have

$$|II| = \left| \int_{x-z<-n, y-z>n} e^{-i(e^z - e^{2y-z})} \frac{dz}{(x-z)(y-z)} \right| \leq Ce^{-y}. \tag{24}$$

In addition,

$$III = \int_{y-z<-n} \frac{dz}{(x-z)(y-z)} = \frac{\ln(1 + \frac{y-x}{n})}{y-x}. \tag{25}$$

By (23)-(25), for $x \in (-ce^y, -n^2)$ and $y > 0$, we have

$$L(x,y) \geq \frac{1}{2} \cdot \frac{\ln(1 + \frac{y-x}{n})}{y-x} \geq \frac{\ln(y-x)}{4(y-x)}. \tag{26}$$

Now, taking $m > 2n^2$ and $f_m = \chi_{(m,2m)}$, we have that for $x \in (-ce^m, -n^2)$,

$$\begin{aligned} L(f_m)(x) &= \int_m^{2m} L(x,y)dy \geq \int_m^{2m} \frac{\ln(y-x)}{4(y-x)}dy = \int_{m+|x|}^{2m+|x|} \frac{\ln y}{4y}dy \\ &= \frac{1}{4}\ln((2m+|x|)(m+|x|))\ln(1+\frac{m}{m+|x|}) \\ &\geq C\cdot \begin{cases} \ln m & \text{for} \quad |x|<m \\ \frac{m\ln|x|}{|x|} & \text{for} \quad |x|\geq m. \end{cases} \end{aligned} \tag{27}$$

Therefore,

$$\|L(f_m)\|_2 > \left(\int_{-m}^{-n^2} |L(f_m)(x)|^2\,dx\right)^{1/2} \geq Cm^{1/2}\ln m = C\ln m\,\|f_m\|_2$$

which means that L is not L^2-bounded. Theorem 3 is proved.

3.2 Proof of Theorem 4

By Fourier transform and Plancherel's formula, we have (see [8] p116)

$$\|H_{P,\gamma}\|_{L^2(R^2)\to L^2(R^2)} \leq \sup_{u\in R^1} \|S_u\|_{L^2(R^1)\to L^2(R^1)} \tag{28}$$

where

$$S_u(f)(x) = p.v. \int_{-\infty}^{+\infty} e^{-iuP(x)\gamma(y)} f(x-y)\frac{dy}{y}. \tag{29}$$

So, to prove Theorem 4, we only need to prove that

$$\|S_u\|_{L^2(R^1)\to L^2(R^1)} \text{ is finite and depends only on } \lambda,\ M \text{ and } \deg(P). \tag{30}$$

To be convenient, let $\deg(P) = -1$ if $P \equiv 0$.

For the case $\deg(P) = -1$, S_u is the usual Hilbert transform. So, (30) holds. Suppose that (30) hold for $\deg(P) < n$ (inductive hypothesis). We shall prove that (30) holds for all P with $\deg(P) = n$.

Now, suppose that $\deg(P) = n$ and P's coeffecient of the term of the highest order be s_0. Take ω_0 such that

$$|s_0 u|\,\omega_0^n\gamma(\omega_0) = 1,$$

and set

$$\widetilde{S}(f)(x) = p.v. \int_{R^1} e^{-is_0u\omega_0^n\gamma(\omega_0)\frac{P(\omega_0 x)}{\omega_0}\frac{\gamma(\omega_0 y)}{\gamma(\omega_0)}} f(x-y)\frac{dy}{y}.$$

Then, $S_u(f)(x) = \widetilde{S}(f_{\omega_0})(\frac{x}{\omega_0})$ where $f_{\omega_0}(x) = f(\omega_0 x)$. Obviously, $\|S_u\|_{L^2(R^1)\to L^2(R^1)} = \left\|\widetilde{S}\right\|_{L^2(R^1)\to L^2(R^1)}$. Noting that $|s_0 u|\,\omega_0^n \gamma(\omega_0) = 1$, to prove (30) with $\deg(P) = n$, we only need to show that

$$\|S\|_{L^2(R^1)\to L^2(R^1)} \text{ is finite and depends only on } \lambda,\ M \text{ and } n \tag{31}$$

where

$$S(f)(x) = p.v.\int_{R^1} e^{iP(x)\gamma(y)} f(x-y)\frac{dy}{y}$$

and P's coeffecient of the term of the highest order is 1, $\gamma(1) = 1$.

Decompose S into two parts

$$\begin{aligned} S(f)(x) &= \left(p.v.\textstyle\int_{-1}^{1} + \sum_{k\geq 0}\int_{2^k\leq|y|\leq 2^{k+1}}\right) e^{iP(x)\gamma(y)} f(x-y)\frac{dy}{y} \\ &= \overline{S}(f)(x) + \textstyle\sum_{k\geq 0} S^{(k)}(f)(x). \end{aligned} \tag{32}$$

Step 1 *We first have*

$$\left\|\overline{S}\right\|_{L^2(R^1)\to L^2(R^1)} \leq C_{\lambda,M,n}. \tag{33}$$

To do so, we make decomposition $f = \sum_j f_j$ where $f_j = f\chi_{I_j}$ and $I_j = [2j-1, 2j+1)$. For any x, $\#\{j : \overline{S}(f)(x) \neq 0\} \leq 2$, so

$$\left|\overline{S}(f)(x)\right|^2 \leq 2\sum_j \left|\overline{S}(f_j)(x)\right|^2. \tag{34}$$

Set $Q_j(x) = P(x) - (x-2j)^n$, and

$$\overline{S}_{Q_j}(f)(x) = p.v.\int_{-1}^{1} e^{iQ_j(x)\gamma(y)} f(x-y)\frac{dy}{y}.$$

By inductive hypothesis, Lemma 5 and the fact that $\deg(Q_j) \leq n-1$, we have

$$\left\|\overline{S}_{Q_j}\right\|_{L^2(R^1)\to L^2(R^1)} \leq C_{\lambda,M,n}. \tag{35}$$

In addition,

$$\begin{aligned} \left|\overline{S}_{Q_j}(f_j)(x) - \overline{S}(f_j)(x)\right| &\leq \textstyle\int_{-1}^{1}\left|e^{iQ_j(x)\gamma(y)} - e^{iP(x)\gamma(y)}\right| |f_j(x-y)|\,\frac{dy}{y} \\ &\leq \textstyle\int_{-1}^{1}\left|\frac{\gamma(y)}{y}\right| |f_j(x-y)|\,dy \leq C_\gamma \int_{R^1}|f_j(x-y)|\,dy \end{aligned} \tag{36}$$

because $\gamma(y) \leq |y|$ for $|y| \leq 1$ Combining (34)-(36), we get

$$\begin{aligned} \left\|\overline{S}(f)\right\|_2 &\leq \left(2\textstyle\int_{-1}^{1}\sum_j \left|\overline{S}(f_j)(x)\right|^2 dx\right)^{1/2} \\ &\leq \left(2C_{\lambda,M,n}^2 \textstyle\int_{R^1}\sum_j |f_j(x)|^2\,dx\right)^{1/2} \leq \sqrt{2}C_{\lambda,M,n}\,\|f\|_2\,. \end{aligned}$$

So, (33) holds.

Step 2 *There is $\epsilon' = \epsilon'(M,n) > 0$, such that*

$$\|L_\mu\|_{L^2(R^1)\to L^2(R^1)} \le C_{\lambda,M,n}\mu^{-\epsilon'} \tag{37}$$

where $\mu \in R^1$ and

$$\begin{array}{l} L_\mu(f)(x) = \int_{R^1} L_\mu(x,y)f(y)dy \\ L_\mu(x,y) = \int_{1\le x-z\le y-z<2} e^{i\mu(P(x)\gamma(x-z)-P(y)\gamma(y-z))}\frac{dz}{(x-z)(y-z)}. \end{array} \tag{38}$$

To do so, let $\varphi(x,y,z) = P(x)\gamma(x-z) - P(y)\gamma(y-z)$, and $U = \{Rez : P(z) = 0\}$, $U_\delta^y = \{x \in R^1 : d(x,U) \le \delta \text{ or } y - x \le \delta\}$ where δ is to be determined. It is easy to see that $|P(x)| \ge \delta^n$ for $x \notin U_\delta^y$. Obviously, $|L_\mu(x,y)| \le 1$, so

$$\sup_y \int_{U_\delta^y} |L_\mu(x,y)|\, dx \le (2n+1)\delta \le C_n\delta. \tag{39}$$

In addition, for $z < x < y$, $\frac{\gamma'(x-z)}{\gamma'(y-z)}$ is strictly decreasing on $z \in (-\infty, x)$, so, there is at most one z' such that

$$\frac{\gamma'(x-z')}{\gamma'(y-z')} = \frac{P(y)}{P(x)}.$$

Now, to be convevient, we may assume that $z' = -\infty$ for the case that $\frac{P(y)}{P(x)} \ge \lim_{z\to-\infty} \frac{\gamma'(x-z)}{\gamma'(y-z)}$, and $z' = x$ for the case that $\frac{P(y)}{P(x)} \le \lim_{z\to x-0} \frac{\gamma'(x-z)}{\gamma'(y-z)} = 0$. And let $B_\delta = \{z \in \mathbf{C} : |z - z'| \le \delta\}$. For $z \notin B_\delta$, let z" be the point in $\overline{zz'}$ such that $d(z, z") = \delta$. For $1 \le x - z \le y - z < 2$, $x \notin U_\delta^y$ and $z \notin B_\delta$, we have

$$\begin{array}{ll} \left|\frac{\varphi_z'(x,y,z)}{P(x)\gamma'(y-z)}\right| & = \left|\frac{\gamma'(x-z)}{\gamma'(y-z)} - \frac{P(y)}{P(x)}\right| \ge \left|\frac{\gamma'(x-z)}{\gamma'(y-z)} - \frac{\gamma'(x-z')}{\gamma'(y-z')}\right| \\ & \ge \left|\frac{\gamma'(x-z)}{\gamma'(y-z)} - \frac{\gamma'(x-z")}{\gamma'(y-z")}\right| \ge c_{\lambda,M}\frac{(y-x)^M\delta}{(2+\delta)^{M+1}}\frac{\gamma'(x-z)}{\gamma'(y-z)} \\ & \ge c_{\lambda,M}\delta^{M+1}\frac{\gamma'(x-z)}{\gamma'(y-z)} \end{array} \tag{40}$$

Thus, for $1 \le x - z \le y - z < 2$, $x \notin U_\delta^y$ and $z \notin B_\delta$, we have

$$|\varphi_z'(x,y,z)| \ge c_{\lambda,M}\delta^{M+1}\gamma'(x-z)\,|P(x)|$$

which means that for $1 \le x - z \le y - z < 2$, $x \notin U_\delta^y$, $z \notin B_\delta$,

$$\begin{array}{l} |\varphi_z'(x,y,z)| \ge c_{\lambda,M}\delta^{M+1+n} \text{ (because } \gamma'(1) \ge 1) \\ |\varphi_z'(x,y,z)| \ge c_{\lambda,M,n}\delta^{M+1} \cdot \left\{ \begin{array}{l} \gamma'(x-z)\,|P(x)| \\ \gamma'(y-z)\,|P(y)|\,. \end{array} \right. \end{array} \tag{41}$$

Therefore,

$$\begin{array}{ll} \frac{|\varphi_{zz}''(x,y,z)|}{|\varphi_z'(x,y,z)|^2} & \le \frac{\gamma''(x-z)|P(x)|}{|\varphi_z'(x,y,z)|^2} + \frac{\gamma''(y-z)|P(y)|}{|\varphi_z'(x,y,z)|^2} \\ & \le C_{\lambda,M,n}\delta^{-2(M+1)}\left(\frac{\gamma''(x-z)}{|\gamma'(x-z)|^2|P(x)|} + \frac{\gamma''(y-z)}{\gamma'(x-z)\gamma'(y-z)|P(x)|}\right) \\ & \le C_{\lambda,M,n}\delta^{-2(M+1)-n}\frac{\gamma''(x-z)}{\gamma'(x-z)} \end{array} \tag{42}$$

because $\frac{\gamma''(y-z)}{\gamma'(y-z)} \leq \frac{\gamma''(x-z)}{\gamma'(x-z)}$.

For fixed x and y, $x \notin U_\delta^y$, $\{z : 1 \leq x - z \leq y - z < 2\} - B_\delta$ consists of at most two intervals. To be convenient, we assume that it consists of one interval Δ. By (41) and (42), we have

$$\begin{aligned} |L_\mu(x,y)| \leq & \left| \int_\Delta e^{i\mu\varphi} \frac{dz}{(x-z)(y-z)} \right| + 2\delta \\ \leq & \frac{1}{\mu} \left. \frac{e^{i\mu\varphi}}{(x-z)(y-z)\varphi'_z} \right|_{\partial\Delta} + \frac{1}{\mu} \left| \int_\Delta e^{i\mu\varphi} \frac{1}{\varphi'_z} \frac{\partial}{\partial z} \left(\frac{1}{(x-z)(y-z)}\right) dz \right| \\ & + \frac{1}{\mu} \left| \int_\Delta e^{i\mu\varphi} \frac{\varphi''_{zz}}{(\varphi'_z)^2} \frac{1}{(x-z)(y-z)} dz \right| + 2\delta \\ \leq & \frac{C}{\mu} \left(\sup_{z\in\overline{\Delta}} |\varphi'_z|^{-1} + \int_\Delta \frac{|\varphi''_{zz}|}{|\varphi'_z|^2} dz \right) + 2\delta \\ \leq & \frac{C_{\lambda,M,n}}{\mu} \left(\delta^{-1-n-M} + \delta^{-2-n-2M} \int_\Delta \frac{\gamma''(x-z)}{|\gamma'(x-z)|^2} dz \right) + 2\delta \\ \leq & \frac{C_{\lambda,M,n}}{\mu} \delta^{-2-n-2M} + 2\delta \end{aligned} \tag{43}$$

for $\gamma'(1) \geq 1$ and γ' is increasing. Noting that $L_\mu(x,y) = 0$ for $y - x > 1$, by (39) and (43), we have

$$\sup_y \int_{R^1} |L_\mu(x,y)|\, dx \leq C_{\lambda,M,n} \left(\frac{\delta^{-2-n-2M}}{\mu} + \delta \right).$$

Taking $\delta = \mu^{-\epsilon}$ with $\epsilon = \frac{1}{2M+n+3}$, we get

$$\sup_y \int_{R^1} |L_\mu(x,y)|\, dx \leq C_{\lambda,M,n} \mu^{-\epsilon}$$

which means that $\|L_\mu\|_{L^1(R^1)\to L^1(R^1)} \leq C_{\lambda,M,n}\mu^{-\epsilon}$. On the other hand, it is obvious that $\|L_\mu\|_{L^\infty(R^1)\to L^\infty(R^1)} \leq C$. So, by Marcinkiewicz interpolation theorem, there is $\epsilon' = \epsilon'(M,n)$ such that (37) holds.

Step 3 *We have*

$$\|R_\mu\|_{L^2(R^1)\to L^2(R^1)} \leq C_{\lambda,M,n} \mu^{-\frac{\epsilon'}{2}} \tag{44}$$

where

$$R_\mu(f)(x) = \int_{1\leq x-y<2} e^{i\mu P(x)\gamma(x-y)} f(y) \frac{dy}{x-y}.$$

Actually, $\|R_\mu\|_{L^2(R^1)\to L^2(R^1)} = \left\|R_\mu \circ R_\mu^*\right\|^{1/2}_{L^2(R^1)\to L^2(R^1)}$, and the kernel of $R_\mu \circ R_\mu^*$ is

$$R_\mu \circ R_\mu^*(x,y) = \int_{1\leq x-z\leq 2, 1\leq y-z\leq 2} e^{i\mu(P(x)\gamma(x-z) - P(y)\gamma(y-z))} \frac{dz}{(x-z)(y-z)}.$$

Note that $R_\mu \circ R_\mu^*(x,y) = L_\mu(x,y) + \overline{L_\mu(y,x)}$ for $x \neq y$, $R_\mu \circ R_\mu^*(x,y) = L_\mu(x,y) = \frac{1}{2}$ for $x = y$. So,

$$\left\|R_\mu \circ R_\mu^*\right\|_{L^2(R^1)\to L^2(R^1)} \leq 2\, \|L_\mu\|_{L^2(R^1)\to L^2(R^1)}. \tag{45}$$

By (37) and (45), we get (44).

Now, by the oddness or evenness of γ, we have

$$\left\|S^{(k)}\right\|_{L^2(R^1)\to L^2(R^1)} \le \left\|S_+^{(k)}\right\|_{L^2(R^1)\to L^2(R^1)}$$

where

$$S_+^{(k)}(f)(x) = \int_{2^k\le x-y\le 2^{k+1}} e^{iP(x)\gamma(x-y)} f(y)\frac{dy}{x-y}.$$

Note that for $x' = 2^{-k}x$,

$$S_+^{(k)}(f)(x) = \int_{1\le x'-y'\le 2} e^{i2^{kn}\gamma(2^k)\frac{P(2^k x')}{2^{kn}}\frac{\gamma(2^k(x'-y'))}{\gamma(2^k)}} f(2^k y')\frac{dy'}{x'-y'} = R_{\mu_k}(f_{2^k})(\frac{x}{2^k})$$

where $f_{2^k}(x) = f(2^k x)$. So,

$$\left\|S_+^{(k)}\right\|_{L^2(R^1)\to L^2(R^1)} = \|R_{\mu_k}\|_{L^2(R^1)\to L^2(R^1)} \le C_{\lambda,M,n}\mu_k^{-\frac{\epsilon'}{2}}$$

where $\mu_k = 2^{kn}\gamma(2^k)$. Therefore,

$$\sum_{k\ge 0}\left\|S^{(k)}\right\|_{L^2(R^1)\to L^2(R^1)} \le \sum_{k\ge 0} C_{\lambda,M,n}(2^{kn}\gamma(2^k))^{-\frac{\epsilon'}{2}} \le C'_{\lambda,M,n}. \qquad (46)$$

From (32)-(33) and (46), we get (31). Theorem 4 is proved now.

References

[1] J. M. Bennett, *Hilbert transforms and maximal functions along variable flat plane curves*, **Trans. Amer. Math. Soc.** 354(2002), 4871-4892.

[2] A. Carbery, M. Christ, J. Vance, S. Wainger and D. K. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, **Duke Math. J.** 59(1989), 675-700.

[3] A. Carbery, A. Seeger, S. Wainger and J. Wright, *Claases of singular operators along variable lines*, **J. Geom. Anal.** 9(1999), 583-605.

[4] A. Carbery and S. Pérez, *Maximal functions and Hilbert transforms along variable flat curves*, **Math. Res. Lett.** 6(1999), 237-249.

[5] A. Carbery, S. Wainger and J. Wright, *Hilbert transforms and maximal functions along variable flat plane curves*, **J. Fourier Anal. Appl.** Special Issue(1995), 119-139.

[6] A. Carbery, S. Wainger and J. Wright, *Hilbert transforms and maximal functions along variable flat plane curves on the Heisengerg group*, **J. Amer. Math. Soc.** 8(1995), 141-179.

[7] A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Hilbert transform for convex curves*, **Duke Math. J.** 50(1983), 735-744.

[8] D. H. Phone and E. M. Stein, *Hilbert integrals, singular integrals and Randon transforms I*, **Acta Math.** 157(1986), 99-157.

[9] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I: Oscillatory integrals*, **J. Funct. Anal.** 73(1987), 179-194.

[10] A. Seeger, *L^2-estimates for a class of singular oscillatory integrals*, **Math. Res. Lett.** 1(1994), 65-73.

[11] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, **Princeton Univ. Press**, Princeton, 1993.

[12] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, **Bull. Amer. Math. Soc.** 84(1978), 1239-1295.