ON A PROOF OF A CONJECTURE OF MARINO-VAFA ON HODGE INTEGRALS

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Abstract. We outline a proof of a remarkable formula for Hodge integrals conjectured by Mariño and Vafa [23] based on large N duality.

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber of over $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i} C$ at the i-th marked point x_i. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of the Hodge bundle.

Hodge integrals arise in the calculations of Gromov-Witten invariants by localization techniques [14, 7]. The explicit evaluation of Hodge integrals is a difficult problem. The Hodge integrals involving only ψ classes can be computed recursively by Witten's conjecture [26] proven by Kontsevich [13]. Algorithms of computing Hodge integrals are described in [2].

In [23], M. Mariño and C. Vafa obtained a closed formula for a generating function of certain open Gromov-Witten invariants, some of which has been reduced to Hodge integrals by localization techniques which are not fully clarified mathematically. This leads to a conjectural formula of Hodge integrals. To state this formula, we introduce some notation, following [28]. Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u + \cdots + (-1)^g \lambda_g$$

be the Chern polynomial of $\mathbb{E}^\vee$, the dual of the Hodge bundle. For a partition μ given by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0,$$

let $|\mu| = \sum_{i=1}^{l(\mu)} \mu_i$, and define

$$\begin{aligned} \mathcal{C}_{g,\mu}(\lambda;\tau) &= -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \\ &\quad \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}, \\ \mathcal{C}_\mu(\lambda;\tau) &= \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}\mathcal{C}_{g,\mu}(\lambda;\tau) \end{aligned}$$

Note that

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} > 0)$. Define generating functions

$$\begin{aligned} \mathcal{C}(\lambda;\tau;p) &= \sum_{|\mu|\geq 1}\mathcal{C}_\mu(\lambda;\tau)p_\mu, \\ \mathcal{C}(\lambda;\tau;p)^\bullet &= e^{\mathcal{C}(\lambda;\tau;p)}. \end{aligned}$$

As pointed out in [23], by comparing computations in [23] with computations in [12], one obtains a conjectural formula for $\mathcal{C}_\mu(\tau)$. This formula is explicitly written down in [28].

(1)

$$\mathcal{C}(\lambda;\tau;p) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_\mu\left(\sum_{\cup_{i=1}^n\mu^i=\mu}\prod_{i=1}^n\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}V_{\nu^i}(\lambda)\right)p_\mu,$$

(2)
$$\mathcal{C}(\lambda;\tau;p)^\bullet = \sum_{|\mu|\geq 0}\left(\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda)\right)p_\mu,$$

where

(3)
$$\begin{aligned} V_\nu(\lambda) = &\prod_{1\leq a<b\leq l(\nu)}\frac{\sin[(\nu_a-\nu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]} \\ &\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\nu_i}2\sin[(v-i+l(\nu))\lambda/2]}. \end{aligned}$$

We now explain the notation on the right-hand sides of (1) and (2). For a partition μ given by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0,$$

χ_μ denotes the character of the irreducible representation of S_d indexed by μ, where $d = |\mu| = \sum_{i=1}^{l(\mu)}\mu_i$. The number κ_μ is defined by

$$\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i).$$

For each positive integer i,

$$m_i(\mu) = |\{j : \mu_j = i\}|.$$

Denote by $C(\nu)$ the conjugacy class of S_d corresponding to the partition ν, and by $\chi_\mu(C(\nu))$ the value of the character χ_μ on the conjugacy class $C(\nu)$. Finally,

$$z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}.$$

In this paper, we will call (1) the Mariño-Vafa formula.

The third author proved in [27] some special cases of the Mariño-Vafa formula and found several interesting applications. He showed in [29] that calculation of BPS numbers in the local $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ geometries can be reduced to the Mariño-Vafa formula, and proved in [30] a special case of a conjecture by A. Iqbal [11] assuming the Mariño-Vafa formula.

We now describe our approach to the Mariño-Vafa formula (1). Denote the right-hand sides of (1) and (2) by $R(\lambda;\tau;p)$ and $R(\lambda;\tau;p)^\bullet$ respectively. In [28], the third author proved the following two equivalent cut-and-join equations similar to the one satisfied by Hurwitz numbers [6], [20], [10, Section 15.2].

Theorem 1.

$$\frac{\partial R}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left(ijp_{i+j} \frac{\partial^2 R}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial R}{\partial p_i} \frac{\partial R}{\partial p_j} + (i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} \right), \tag{4}$$

$$\frac{\partial R^\bullet}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left(ijp_{i+j} \frac{\partial^2 R^\bullet}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial R^\bullet}{\partial p_{i+j}} \right). \tag{5}$$

Here is a crucial observation: One can rewrite (5) as a sequence of systems of ordinary equations, one for each positive integer d, hence if $\mathcal{C}(\lambda;\tau;p)^\bullet$ satisfies (5), then it is determined by the initial value $\mathcal{C}(\lambda;0;p)^\bullet$. To prove (1) or (2), it suffices to prove the following two statements:

(a) Equation (4) is satisfied by $\mathcal{C}(\lambda;\tau;p)$.
(b) $\mathcal{C}(\lambda;0;p) = R(\lambda;0;p)$.

Or equivalently,

(a)' Equation (5) is satisfied by $\mathcal{C}(\lambda;\tau;p)^\bullet$.
(b)' $\mathcal{C}(\lambda;0;p)^\bullet = R(\lambda;0;p)^\bullet$.

It is shown in [28] that (b) holds. Therefore, the Mariño-Vafa formula (1) follows from the following theorem.

Theorem 2.

$$\frac{\partial \mathcal{C}}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left(ijp_{i+j} \frac{\partial^2 \mathcal{C}}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \mathcal{C}}{\partial p_i} \frac{\partial \mathcal{C}}{\partial p_j} + (i+j)p_i p_j \frac{\partial \mathcal{C}}{\partial p_{i+j}} \right) \tag{6}$$

The rest of the paper is organized as follows. In Section 2, we give a proof of the initial condition (b). In Section 3, we give the proof of Theorem 1 in [28]. In Section 4, we outline the proof of Theorem 2 in [22]. The details we omit here are straightforward calculations which will be given in [22]. Complete lists of relevant references will be given in [28, 22].

2. Initial Condition

The proof of the initial condition (b) needs the following two theorems.

Theorem 2.1. *We have*

$$\mathcal{C}(\lambda;0;p) = -\sum_{n>0} \frac{\sqrt{-1}^{n+1} p_n}{2n\sin(n\lambda/2)}. \tag{7}$$

Proof. When $l(\mu) > 1$, we clearly have

$$\mathcal{C}_\mu(\lambda;0) = 0.$$

When $\mu = (n)$ we have

$$\begin{aligned}
\mathcal{C}_{(n)}(\lambda;0) &= -\sum_{g\geq 0} \lambda^{2g-1}\sqrt{-1}^{n+1} \frac{\prod_{a=1}^{n-1}(n\cdot 0 + a)}{(n-1)!} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(0)\Lambda_g^\vee(-1)}{1-n\psi_1} \\
&= -\frac{\sqrt{-1}^{n+1}}{n} \sum_{g\geq 0} (n\lambda)^{2g-1} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2} \\
&= -\frac{\sqrt{-1}^{n+1}}{n^2\lambda} \cdot \frac{n\lambda/2}{\sin(n\lambda/2)} \\
&= -\frac{\sqrt{-1}^{n+1}}{2n\sin(n\lambda/2)}.
\end{aligned}$$

In the second equality we have used the Mumford's relations [24, 5.4]:

$$\Lambda_g^\vee(1)\Lambda_g^\vee(-1) = (-1)^g.$$

In the third equality we have used [3, Theorem 2]. This proves (7). □

Theorem 2.2. *We have the following identity:*

$$\log\left(\sum_{n\geq 0} \sum_{|\rho|=n} \frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta \right) = -\sum_{d\geq 1} \frac{\sqrt{-1}^{d+1} p_d}{2d\sin(d\lambda/2)}. \tag{8}$$

For a partition η,

$$n(\eta) = \sum_i (i-1)\eta_i = \sum_i \binom{\eta_i'}{2}.$$

For any box $e \in \eta$, denote by $h(e)$ its hook length. Then

$$\sum_{x\in\eta} h(x) = n(\eta) + n(\eta') + |\eta|.$$

Lemma 2.1. *Introducing formal variables $x_1, \dots, x_n, \dots$ such that*

$$p_i(x_1, \dots, x_n, \dots) = x_1^i + \cdots + x_n^i + \cdots.$$

Then for for any positive integer n, we have

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta = \frac{1}{\prod_{i,j}(1-tx_iq^{j-1})}. \tag{9}$$

Proof. Recall the following facts about Schur polynomials:

$$s_\rho(x) = \sum_\eta \frac{\chi_\rho(\eta)}{z_\eta} p_\eta(x), \tag{10}$$

$$s_\rho(1, q, q^2, \dots) = \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})}, \tag{11}$$

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} s_\rho(x)s_\rho(y) = \frac{1}{\prod_{i,j}(1-tx_iy_j)}. \tag{12}$$

Combining these two identities, one gets:

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} s_\rho(x) = \frac{1}{\prod_{i,j}(1-tx_iq^{j-1})}.$$

The proof is completed by (10). □

As a corollary we now prove (8). First we need the following:

Lemma 2.2. *For any partition ρ we have*

$$\frac{1}{2}\sum_{e\in\rho} h(e) - n(\rho) = \frac{1}{4}\kappa_\rho + \frac{1}{2}|\rho|. \tag{13}$$

Proof.

$$\begin{aligned} \frac{1}{2}\sum_{e\in\rho} h(e) - n(\rho) &= \frac{1}{2}(n(\rho') - n(\rho) + |\rho|) \\ &= \frac{1}{2}(\sum_i \binom{\rho_i}{2} - \sum_i (i-1)\rho_i + |\rho|) \\ &= \frac{1}{4}(\sum_i \rho_i(\rho_i - 1) - 2\sum_i i\rho_i + 4|\rho|) \\ &= \frac{1}{4}\kappa_\rho + \frac{1}{2}|\rho|. \end{aligned}$$

□

Let $q = e^{-\sqrt{-1}\lambda}$, and $t = \sqrt{-1}q^{1/2}$, then we have

$$\begin{aligned} &\sum_{n\geq 0} t^n \sum_{|\rho|=n} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\ = &\sum_{n\geq 0} \sqrt{-1}^n q^{n/2} \sum_{|\rho|=n} \frac{q^{n(\rho)-\frac{1}{2}\sum_{e\in\rho} h(e)}}{\prod_{e\in\rho}(q^{-h(e)/2} - q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\ = &\sum_{n\geq 0} \sqrt{-1}^n q^{n/2} \sum_{|\rho|=n} \frac{q^{-\frac{1}{4}\kappa_\rho - \frac{1}{2}n}}{\prod_{e\in\rho}(q^{-h(e)/2} - q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\ = &\sum_{n\geq 0} \sum_{|\rho|=n} \frac{e^{\frac{1}{2}f_\rho(2)\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta. \end{aligned}$$

Hence by (9),

$$
\begin{aligned}
& \log\left(\sum_{n\geq 0}\sum_{|\rho|=n}\frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho}2\sin(h(e)\lambda/2)}\frac{\chi_\rho(\eta)}{z_\rho}p_\eta\right)\\
= & \log\left(\sum_{n\geq 0}t^n\sum_{|\rho|=n}\frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})}\frac{\chi_\rho(\eta)}{z_\rho}p_\eta\right)\\
= & \log\frac{1}{\prod_{i,j}(1-tx_iq^{j-1})} = \sum_{i,j\geq 1}\sum_{d\geq 1}\frac{1}{d}t^dq^{d(j-1)}x_i^d\\
= & \sum_{j\geq 1}\sum_{d\geq 1}\frac{1}{d}t^dq^{d(j-1)}p_d = \sum_{d\geq 1}\frac{p_d}{d}\frac{t^d}{1-q^d}\\
= & -\sum_{d\geq 0}\frac{\sqrt{-1}^{d+1}p_d}{2d\sin(d\lambda/2)}.
\end{aligned}
$$

By (8) we have

$$
R(\lambda;0;p) = \log\left(\sum_{n\geq 0}\sum_{|\rho|=n}\frac{\chi_\rho(\eta)}{z_\eta}e^{\frac{1}{4}\kappa_\rho\lambda}V_\rho p_\eta\right) = -\sum_{d\geq 1}\frac{\sqrt{-1}^{d+1}p_d}{2d\sin(d\lambda/2)},
$$

where we have used the following identity proved in [28]:

$$
V_\mu = \frac{1}{2^l\prod_{x\in\mu}\sin[h(x)\lambda/2]}.
$$

Hence (b) is proved.

3. Proof of Theorem 1

Recall

$$
c_\mu = \sum_{g\in C_\mu} g
$$

lies in the center of the group algebra $\mathbb{C}S_d$, hence it acts as a scalar $f_\nu(\mu)$ on any irreducible representation R_ν. In other words, let $\rho : S_d \to \operatorname{End} R_\nu$ be the representation indexed by ν, then

$$
\sum_{g\in C(\mu)}\rho_\nu(g) = f_\nu(\mu)\,\mathrm{id}\,.
$$

We need the following interpretation of κ_ν in terms of character:

$$
\kappa_\nu = 2f_\nu(C(2)).
$$

See e.g. [25, (5)]. Here we use $C(2)$ to denote the class of transpositions. We need the following result:

Lemma 3.1. *Suppose $h \in S_d$ has cycle type μ. The product $C_{(2)}\cdot h$ is a sum of elements of S_d whose type is either a cut or a join of μ. More precisely, there are $ijm_i(\mu)m_j(\mu)$ elements obtained from h by joining an i-cycle in h to a j-cycle in h, and there are $(i+j)m_{i+j}(\mu)$ elements obtained from h by cutting an $(i+j)$-cycle into an i-cycle and a j-cycle.*

Proof. Denote by $[s_1, \dots, s_k]$ a k-cycle. Then

$$[s,t] \cdot [s, s_2, \dots, s_i, t, t_2, \dots, t_j] = [s, s_2, \dots, s_i][t, t_2, \dots, t_j],$$

i.e., an $(i+j)$-cycle is cut into an i-cycle and a j-cycle. Conversely,

$$[s,t] \cdot [s, s_2, \dots, s_i][t, t_2, \dots, t_j] = [s, s_2, \dots, s_i, t, t_2, \dots, t_j],$$

i.e., an i-cycle and a j-cycle is joined to an $(i+j)$-cycle. Hence for a permutation h of type μ, $c_{(2)} \cdot h$ is a sum of all elements obtained from h by either a cut or a join. Fix a pair of i-cycle and j-cycle of h, there are $i \cdot j$ different ways to join them to an $(i+j)$-cycle. Taking into the account of $m_i(\mu)$ choices of i-cycles, and $m_j(\mu)$ choices of j-cycles, we get

$$ijm_i(\mu)m_j(\mu)$$

different ways to obtain an element from h by joining an i-cycle in h to a j-cycle in h. Similarly, fix an $(i+j)$-cycle of h, there are $i+j$ different ways to cut it into an i-cycle and a disjoint j-cycle in h. And taking into account the number of $(i+j)$-cycles in h, we get

$$(i+j)m_{i+j}(\mu)$$

different ways to obtain an element from h by cutting an $(i+j)$-cycle into an i-cycle and a j-cycle. □

Now we have for any $h \in S_d$ of cycle type μ

$$\begin{aligned}
& \sum_\mu f_\nu(2) \frac{\chi_\nu(\mu)}{z_\mu} p_\mu \\
= & \sum_\mu \mathrm{tr}[f_\nu(2)\, \mathrm{id} \cdot \rho_\nu(h)] \cdot \prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)} m_i(\mu)!} \\
= & \sum_\mu \mathrm{tr}[\sum_{g \in C(2)} \rho_\nu(g) \cdot \rho_\nu(h)] \cdot \prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)} m_i(\mu)!} \\
= & \sum_\mu \mathrm{tr}\, \rho_\nu(\sum_{g \in C(2)} g \cdot h) \cdot \prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)} m_i(\mu)!} \\
= & \sum_\mu \left(\sum_{\eta \in J_{i,j}(\mu)} ijm_i(\mu)m_j(\mu)\chi_\nu(\eta) + \sum_{\eta \in C_{i,j}(\mu)} (i+j)m_{i+j}(\mu)\chi_\nu(\eta) \right) \cdot \prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)} m_i(\mu)!} \\
= & \left(ijp_{i+j} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} \right) \sum_\eta \frac{\chi_\nu(\eta)}{z_\eta} p_\eta.
\end{aligned}$$

Here we have the following notations. Let μ, η be two partitions, both represented by Young diagrams. We write $\eta \in J_{i,j}(\mu)$ and $\mu \in C_{i,j}(\eta)$ if η is obtained from μ by remove a row of length i and a row of length j, then adding a row of length

$i+j$. It follows that

$$\begin{aligned} & \frac{\partial R(\lambda;\tau;p)^\bullet}{\partial \tau} \\ = & \frac{\sqrt{-1}\lambda}{2}\sum_{\mu,\nu}\left(f_\nu(2)\frac{\chi_\nu(C(\mu))}{z_\mu}p_\mu\right)e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda) \\ = & \frac{\sqrt{-1}\lambda}{2}\left(ijp_{i+j}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}+(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}\right)\sum_\eta\frac{\chi_\nu(\eta)}{z_\eta}p_\eta e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda). \end{aligned}$$

This finishes the proof of Theorem 1.

4. Proof of Theorem 2

4.1. **Moduli space of relative morphisms.** We first describe the moduli space of stable relative morphisms to $\mathbb{P}^1$ used in [19]. The moduli spaces of stable relative morphisms are constructed by J. Li [15]. The construction in symplectic geometry was carried out independently by Li-Ruan [18] and Ionel-Parker [9, 10].

Let $\mathbb{P}^1[m]$ be a chain of $m+1$ copies $\mathbb{P}^1$, such that the i-th copy is glued to the $(i+1)$-th copy at the point $p_1^{(i)}$ for $i \leq m$. The first copy will be referred to as the root component, and the other components will be called the bubble components A point $p_1^{(m)}$ is fixed on the $(m+1)$-th component. Denote by $\pi[m] : \mathbb{P}^1[m] \to \mathbb{P}^1$ the map which is identity on the root component and contracts all the bubble components to $p_1^{(0)}$.

Let μ be a partition of $d > 0$. Let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ be the moduli space of morphisms

$$f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m],$$

such that

1. $(C, x_1, \ldots, x_{l(\mu)})$ is a prestable curve of genus g with $l(\mu)$ marked points.
2. $f^{-1}(p_1^{(m)}) = \sum_{i=1}^{l(\mu)}$ as Cartier divisors, and $\deg(\pi[m]\circ f) = d$.
3. The preimage of each node in $\mathbb{P}^1[m]$ consists of nodes of C. If $f(y) = p_1^i$ and C_1 and C_2 are two irreducible components of C which intersects at y, then $f|_{C_1}$ and $f|_{C_2}$ has the same contact order to p_1^i at y.
4. The automorphism group of f is finite.

Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of the bubble components of $\mathbb{P}^1[m]$. In particular, this defines the automorphism group in the stability condition (4) above.

In [15, 16], J. Li showed that $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$r = 2g - 2 + |\mu| + l(\mu),$$

so it has a virtual fundamental class of degree r.

4.2. **Torus action.** Consider the $\mathbb{C}^*$-action

$$t\cdot[z^0 : z^1] = [tz^0 : z^1]$$

on $\mathbb{P}^1$. It has two fixed points $p_0 = [0 : 1]$ and $p_1 = [1 : 0]$. This induces an action on $\mathbb{P}^1[m]$ by the action on the root component induced by the isomorphism to $\mathbb{P}^1$, and the trivial actions on the bubble components. This in turn induces an action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$.

4.3. **The branch morphism.** There is a branch morphism

$$\mathrm{Br}: \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \to \mathrm{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r.$$

Note that $\mathbb{P}^r$ can be identified with $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))$, and the isomorphism

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r)) \cong \mathrm{Sym}^r \mathbb{P}^1$$

is given by $[s] \mapsto \mathrm{div}(s)$. The $\mathbb{C}^*$ action on $\mathbb{P}^1$ induces a $\mathbb{C}^*$ action on $H^0(\mathbb{P}^1, \mathcal{O}(r))$ by

$$t \cdot (z^0)^k (z^1)^{r-k} = t^{-k} (z^0)^k (z^1)^{r-k}.$$

So $\mathbb{C}^*$ acts on $\mathbb{P}^r$ by

$$t \cdot [a_0, a_1, \ldots, a_r] = [a_0, t^{-1} a_1, \ldots, t^{-r} a_r],$$

where $(a_0, a_1, \ldots, a_r)$ corresponds to $\sum_{k=0}^r a_k (z_0)^k (z^1)^{r-k} \in H^0(\mathbb{P}^1, \mathcal{O}(r))$. With this action, the branch morphism is $\mathbb{C}^*$-equivariant.

4.4. **The Obstruction Bundle.** In [19], J. Li and Y. Song constructed an obstruction bundle over the stratum where the target is $\mathbb{P}^1[0] = \mathbb{P}^1$, and proposed an extension over the entire $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. Here we use a different extension which is equivalent to the one used in [1].

Let $\pi[m] : \mathbb{P}^1[m] \to \mathbb{P}^1$ be the contraction to the root component, and denote $\tilde{f} = \pi[m] \circ f$. Dual to the obstruction space at a map $f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m]$, consider the vector bundle V with fiber at f given by

$$H^1(C, \mathcal{O}_C(-D)) \oplus H^1(C, \tilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1)),$$

where $D = x_1 + \ldots + x_{l(\mu)}$. It is a direct sum of two vector bundles V_D and V_{D_d}.

Note that

$$H^0(C, \mathcal{O}_C(-D)) = H^0(C, \tilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

so the ranks of V_D and V_{D_d} are, by Riemann-Roch, $l(\mu) + g - 1$ and $d + g - 1$, respectively.

We lift $\mathbb{C}^*$ action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ to V_D and V_{D_d} as follows. The action on V_{D_d} comes from an action on $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ with weights p and $p+1$ at the two fixed points p_0 and p_1, respectively, where $p \in \mathbb{Z}$. The fiber of V_D does not depend on the map f, so the fibers over two points in the same orbit of the $\mathbb{C}^*$ action can be canonically identified. The action of $\lambda \in \mathbb{C}^*$ on V_D is multiplication by λ^{-p-1}.

4.5. **Functorial localization.** Let $T = \mathbb{C}^*$. We will compute

$$\mathrm{Br}_* e_T(V) = \sum_{l=0}^{r} a_l(p) H^l u^{r-l}.$$

by virtual functorial localization [21].

Let $F(p, x) = \sum_{l=0} a_l(p) x^l$. We have

$$\frac{f_k^* \mathrm{Br}_* e_T(V)}{e_T(T_{p_k} \mathbb{P}^r)} = \frac{F(p,k)}{(-1)^{r-k} k!(r-k)!}.$$

By functorial localization, we have

$$\int_{p_{r-k}} \frac{f_{r-k}^* \mathrm{Br}_* e_T(V)}{e_T(T_{p_{r-k}} \mathbb{P}^r)} = \sum_{F \subset \mathrm{Br}^{-1}(p_{r-k})} \int_{[F]^{vir}} \frac{e_T(V)}{e_T(N_F^{vir})}$$

for $k = 0, \ldots, r$, where N_F is the virtual normal bundle of the fixed loci F in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. It is computed in [22] that

$$\sum_{F\subset \mathrm{Br}^{-1}(p_{r-k})} \int_{[F]^{vir}} \frac{e_T(V)}{e_T(N_F^{vir})} = (p+1)^k J^k_{g,\mu}(p),$$

where $J^k_{g,\mu}(p)$ is a degree $r-k$ polynomial in p, and

$$J^k_{g,\mu}(-p-1) = (-1)^{d-l(\mu)+k} J^k_{g,\mu}(p).$$

Moreover, we have

$$\begin{aligned} J^0_{g,\mu}(p) &= \sqrt{-1}^{l(\mu)-|\mu|} \mathcal{C}_{g,\mu}(p), \\ J^1_{g,\mu}(p) &= \sqrt{-1}^{l(\mu)-|\mu|-1} \Bigg(\sum_{\nu\in J(\mu)} I_1(\nu)\mathcal{C}_{g,\nu}(p) + \sum_{\nu\in C(\mu)} I_2(\nu)\mathcal{C}_{g,\nu}(p) \\ &\quad + \sum_{g_1+g_2=g} \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)\mathcal{C}_{g_1,\nu^1}(p)\mathcal{C}_{g_2,\nu^2}(p) \Bigg). \end{aligned}$$

Here we use the notation in [20]. The set $J(\mu)$ (join) consists of partitions of d of the form

$$\nu = (\mu_1, \ldots, \hat{\mu}_i, \ldots, \hat{\mu}_j, \ldots, \mu_{l(\mu)}, \mu_i + \mu_j)$$

and the set $C(\mu)$ (cut) consists of partitions of d of the form

$$\nu = (\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_{l(\mu)}, j, k)$$

where $j + k = \mu_i$. The precise definitions of I_1, I_2, and I_3 can be found in [20]. It follows from the definition that (6) in Theorem 2 is equivalent to

$$\frac{d}{d\tau} J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau).$$

Since

$$\begin{aligned} & F(p,x) \\ =\ & \sum_{k=0}^{r} \frac{F(p,k)}{(-1)^{r-k} k!(r-k)!} x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-r) \\ =\ & \sum_{k=0}^{r} (p+1)^{r-k} J^{r-k}_{g,\mu}(p) x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-r) \\ =\ & \sum_{k=0}^{r} (p+1)^{k} J^{k}_{g,\mu}(p) x(x-1)\cdots(x-(r-k-1))(x-(r-k+1))\cdots(x-r), \end{aligned}$$

therefore,

$$\mathrm{Br}_* e_T(V) = \sum_{k=0}^{r} (p+1)^k J^k_{g,\mu}(p) H(H-u)\cdots(H-(r-k-1)u)(H-(r-k+1)u)\cdots(H-ru).$$

4.6. **Final Calculations.** Let $\tau = -p-1$, then

$$\begin{aligned}
\mathrm{Br}_* e_T(V) &= \sum_{k=0}^{r} (-\tau)^k J^k_{g,\mu}(-\tau-1) H(H-u)\cdots(H-(r-k-1)u) \\
&\quad \cdot (H-(r-k+1)u)\cdots(H-ru) \\
&= \sum_{k=0}^{r} (-\tau)^k (-1)^{d-l(\mu)+k} J^k_{g,\mu}(\tau) H(H-u)\cdots(H-(r-k-1)u) \\
&\quad \cdot (H-(r-k+1)u)\cdots(H-ru)
\end{aligned}$$

Therefore,

$$\mathrm{Br}_* e_T(V) = (-1)^{d-l(\mu)} \sum_{k=0}^{r} \tau^k J^k_{g,\mu}(\tau) H(H-u)\cdots(H-(r-k-1)u)(H-(r-k+1)u)\cdots(H-ru).$$

For $i = 0, \ldots, r-1$, we have

$$\begin{aligned}
& H^i H(H-u)\cdots(H-(r-k-1)u)(H-(r-k+1)u)\cdots(H-ru) \\
= & ((H-(r-k)u)+(r-k)u)^i H(H-u)\cdots(H-(r-k-1)u) \\
& \cdot (H-(r-k+1)u)\cdots(H-ru) \\
= & ((r-k)u)^i H(H-u)\cdots(H-(r-k-1)u)(H-(r-k+1)u)\cdots(H-ru)
\end{aligned}$$

since

$$H(H-u)\cdots(H-ru) = 0.$$

Therefore,

$$\int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i = (-1)^{d-l(\mu)} u^i \sum_{k=0}^{r} (r-k)^i \tau^k J^k_{g,\mu}(\tau).$$

Let $J^k_{g,\mu}(\tau) = \sum_{j=0}^{r-k} a^k_j \tau^j$. We have

$$u^{-i} \int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i = (-1)^{d-l(\mu)} \sum_{l=0}^{r} \left(\sum_{j+k=l} (r-k)^i a^k_j \right) \tau^l.$$

Here is a crucial observation: as a polynomial in τ, $u^{-i}\int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i$ is of degree no more than i. Therefore,

$$\sum_{j+k=l} (r-k)^i a^k_j = 0$$

for $0 \leq i < l \leq r$. Now fix l such that $1 \leq l \leq r$. We have

$$\sum_{k=0}^{l} (r-k)^i a^k_{l-k} = 0, \quad 0 \leq i < l, \tag{14}$$

which is a system of l linear equations of the $l+1$ variables $\{a^k_{l-k} : k = 0, \ldots, l\}$.

Both

$$\{(r-t)^i : i = 0, \ldots, l-1\}$$

and

$$\{1, t, t(t-1), \ldots, t(t-1)\ldots(t-l+2)\}$$

are bases of the vector space

$$\{f(t) \in \mathbb{Q}[t] : \deg(f) \leq l-1\},$$

so there exists an invertible $l\times l$ matrix $(A_{ij})_{0\leq i,j\leq l-1}$ such that

$$t(t-1)\cdots(t-i+1)=\sum_{j=0}^{l-1}A_{ij}(r-t)^j.$$

In particular,

$$k(k-1)\cdots(k-i+1)=\sum_{j=0}^{l}A_{ij}(r-k)^j.$$

for $k=0,1,\ldots,l$, so (14) is equivalent to

$$\sum_{k=0}^{l}k(k-1)\cdots(k-i+1)a^k_{l-k}=0,\qquad 0\leq i<l,$$

i.e.,

$$\sum_{k=i}^{l}\frac{k!}{(k-i)!}a^k_{l-k}=0,\qquad 0\leq i<l.$$

The above equations can be rewritten as

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1\\ 0 & 1! & 2 & \cdots & \cdots & \cdots & l\\ 0 & 0 & 2! & 3\cdot 2 & \cdots & \cdots & l(l-1)\\ 0 & 0 & 0 & 3! & \cdots & \cdots & l(l-1)(l-2)\\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0 & (l-1)! & l(l-1)\cdots 2 \end{pmatrix}\begin{pmatrix} a^0_l\\ a^1_{l-1}\\ \vdots\\ \vdots\\ a^l_0\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ \vdots\\ \vdots\\ 0\end{pmatrix}.$$

The kernel is clearly one dimensional. One can check that the kernel is given by

$$a^k_{l-k}=(-1)^k\frac{l!}{k!(l-k)!}a^0_l. \tag{15}$$

Note that (15) for $l=1,\ldots,r$ is equivalent to

$$J^k_{g,\mu}(\tau)=\frac{(-1)^k}{k!}\frac{d^k}{d\tau^k}J^0_{g,\mu}(\tau)$$

for $k=0,\ldots,r$. In particular,

$$J^1_{g,\mu}(\tau)=-\frac{d}{d\tau}J^0_{g,\mu}(\tau)$$

which is equivalent to the cut-and-join equation (6) in Theorem 2.

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