A FORMULA OF TWO-PARTITION HODGE INTEGRALS

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1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point x_i. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of the Hodge bundle.

The study of Hodge integrals is an important part of the intersection theory on $\overline{\mathcal{M}}_{g,n}$. Hodge integrals also naturally arise when one computes Gromov-Witten invariants by localization techniques. For example, the following generating series of Hodge integrals arises when one computes local invariants of a toric Fano surface in a Calabi-Yau 3-fold by virtual localization [29]:

$$\begin{aligned} G_{\mu^+,\mu^-}(\lambda;\tau) = & -\frac{(\sqrt{-1}\lambda)^{l(\mu^+)+l(\mu^-)}}{z_{\mu^+}\cdot z_{\mu^-}}\,[\tau(\tau+1)]^{l(\mu^+)+l(\mu^-)-1} \\ & \cdot \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}\left(\mu_i^+\tau + a\right)}{\mu_i^+!} \cdot \prod_{i=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_i^--1}\left(\mu_i^-\frac{1}{\tau} + a\right)}{\mu_i^-!} \\ & \cdot \sum_{g\geq 0} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)} \frac{1}{\mu_i^+}\left(\frac{1}{\mu_i^+} - \psi_i\right) \prod_{j=1}^{l(\mu^-)} \frac{\tau}{\mu_i^-}\left(\frac{\tau}{\mu_j^-} - \psi_{l(\mu^+)+j}\right)}, \end{aligned} \tag{1}$$

where λ, τ are variables, $(\mu^+, \mu^-) \in \mathcal{P}^2_+$, the set of pairs of partitions which are not both empty, and

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

We will call the Hodge integrals in $G_{\mu^+,\mu^-}(\lambda;\tau)$ the *two-partition Hodge integrals.*

The purpose of this paper is to prove the following formula conjectured in [30]:

$$G^\bullet(\lambda; p^+, p^-; \tau) = R^\bullet(\lambda; p^+, p^-; \tau) \tag{2}$$

where

$$G^\bullet(\lambda; p^+, p^-; \tau) = \exp\left(\sum_{(\mu^+,\mu^-)\in\mathcal{P}_+^2} G_{\mu^+,\mu^-}(\lambda;\tau) p^+_{\mu^+} p^-_{\mu^-} \right)$$

$$\begin{aligned} & R^\bullet(\lambda; p^+, p^-; \tau) \\ = & \sum_{|\mu^\pm|=|\nu^\pm|} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^-}(\mu^-)}{z_{\mu^-}} e^{\sqrt{-1}(\kappa_{\nu^+}\tau + \kappa_{\nu^-}\tau^{-1})\lambda/2} \mathcal{W}_{\nu^+,\nu^-}(e^{\sqrt{-1}\lambda}) p^+_{\mu^+} p^-_{\mu^-}, \end{aligned}$$

$p^\pm = (p_1^\pm, p_2^\pm, \ldots)$ are formal variables, and

$$p_\mu^\pm = p_{\mu_1}^\pm \cdots p_{\mu_h}^\pm$$

if $\mu = (\mu_1 \geq \cdots \geq \mu_h > 0)$. See Section 2 for notation in the definition of $R^\bullet(\lambda; p^+, p^-; \tau)$.

Formula (2) is motivated by a formula of one-partition Hodge integrals conjectured by M. Mariño and C. Vafa in [23] and proved by us in [21]. See [25] for another approach to the Mariño-Vafa formula. The Mariño-Vafa formula can be obtained by setting $p^- = 0$ in (2). In a recent paper [4], D.E. Diaconescu and B. Florea conjectured a relation between three-partition Hodge integrals and the topological vertex [1]. A mathematical theory of the topological vertex will be developed in [20].

The generating function $R^\bullet(\lambda; p^+, p^-; \tau)$ is a combinatorial expression involving the representation theory of Kac-Moody Lie algebras. It is also related to the HOMFLY polynomial of the Hopf link and the Chern-Simon theory [26, 24]. In [31], the third author used (2) and a combinatorial trick called the chemistry of $\mathbb{Z}_k$-colored labelled graphs to prove a formula conjectured by A. Iqbal in [12] which expresses the generating function of Gromov-Witten invariants in all genera of local toric Calabi-Yau threefolds in terms of $\mathcal{W}_{\mu,\nu}$. See [12, 1, 4] for surveys of works on this subject.

Our strategy to prove (2) is based on the following cut-and-join equation of $R^\bullet$ observed in [30]:

$$\frac{\partial}{\partial\tau} R^\bullet = \frac{\sqrt{-1}\lambda}{2}(C^+ + J^+)R^\bullet - \frac{\sqrt{-1}\lambda}{2\tau^2}(C^- + J^-)R^\bullet \tag{3}$$

where

$$C^\pm = \sum_{i,j} (i+j) p_i^\pm p_j^\pm \frac{\partial}{\partial p_{i+j}^\pm}, \qquad J^\pm = \sum_{i,j} ij p_{i+j}^\pm \frac{\partial^2}{\partial p_i^\pm \partial p_j^\pm}.$$

Equation (3) can be derived by the method in [28, 21]. In [30], the third author proved that

Theorem 1 (initial values)**.**

$$G^\bullet(\lambda; p^+, p^-; -1) = R^\bullet(\lambda; p^+, p^-; -1). \tag{4}$$

So (2) follows from the main theorem in this paper:

Theorem 2 (cut-and-join equation of $G^\bullet$)**.**

$$\frac{\partial}{\partial\tau} G^\bullet = \frac{\sqrt{-1}\lambda}{2}(C^+ + J^+)G^\bullet - \frac{\sqrt{-1}\lambda}{2\tau^2}(C^- + J^-)G^\bullet \tag{5}$$

Both [21, Theorem 2] (cut-and-join equation of one-partition Hodge integrals) and Theorem 2 are proved by localization method. We compute certain relative Gromov-Witten invariants by virtual localization, and get an expression in terms of one-partition or two-partition Hodge integrals and certain integrals of target ψ classes. In [21], we used functorial localization to push forward calculations to projective spaces, where the equivariant cohomology is completely understood, and derived [21, Theorem 2] without using much information about integrals of target ψ classes. In this paper, we relate integrals of target ψ classes to double Hurwitz numbers, and use properties of double Hurwitz numbers to prove Theorem 2. More precisely, for each $(\mu^+,\mu^-)\in\mathcal{P}^2_+$, we will define a generating function

$$K^\bullet_{\mu^+,\mu^-}(\lambda)$$

of certain relative Gromov-Witten invariants of $\mathbb{P}^1\times\mathbb{P}^1$ blowup at a point, and use localization method to derive the following expression:

(6)

$$K^\bullet_{\mu^+,\mu^-}(\lambda)=\sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(-\sqrt{-1}\tau\lambda)z_{\nu^+}G^\bullet_{\nu^+,\nu^-}(\lambda;\tau)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}(\frac{-\sqrt{-1}}{\tau}\lambda)$$

In (6), $\Phi^\bullet_{\mu,\nu}(\lambda)$ is a generating function of double Hurwitz numbers, and z_μ is defined in Section 2.1. It turns out that (6) is equivalent to the following equation:

$$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)=\sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(\sqrt{-1}\tau\lambda)z_{\nu^+}K^\bullet_{\nu^+,\nu^-}(\lambda)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}(\frac{\sqrt{-1}}{\tau}\lambda) \tag{7}$$

So Theorem 2 (cut-and-join equation of $G^\bullet$) follows from the cut-and-join equations of double Hurwitz numbers. As a consequence, one can compute $K_{\mu^+,\mu^-}(\lambda)$ in terms of $\mathcal{W}_{\nu^+,\nu^-}$ (Corollary 3.5). We will give three derivations of the cut-and-join equations of double Hurwitz numbers: by combinatorics (Section 3.3), by gluing formula (Section 5.4), and by localization (Section 5.8).

The rest of the paper is arranged as follows. In Section 2, we give the precise statement of (2), and recall the proof of Theorem 1 (initial values). In Section 3 we give a combinatorial study of double Hurwitz numbers, and derive Theorem 2 (the cut-and-join equation of $G^\bullet$) from (6) and some identities of double Hurwitz numbers. In Section 4, we review J. Li's works [16, 17] on moduli spaces of relative stable morphisms, and virtual localization on such moduli spaces [9, 11]. In Section 5, we give a geometric study of double Hurwitz numbers. In Section 6, we introduce the geometric objects involved in the proof of (6). In Section 7, we prove (6) by arranging the localization contribution in a neat way.

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2. The Conjecture

2.1. **Partitions.** We recall some notation of partitions. Given a partition

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0),$$

write $l(\mu) = h$, and $|\mu| = \mu_1 + \cdots + \mu_h$. Define

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1).$$

For each positive integer j, define

$$m_j(\mu) = |\{i : \mu_i = j\}|.$$

Then

$$|\operatorname{Aut}(\mu)| = \prod_j (m_j(\mu))!.$$

Define

$$z_\mu = \mu_1 \cdots \mu_{l(\mu)} |\operatorname{Aut}(\mu)| = \prod_j \left(m_j(\mu)! j^{m_j(\mu)} \right).$$

Let $\mathcal{P}$ denote the set of partitions. We allow the empty partition and take

$$l(\emptyset) = |\emptyset| = \kappa_\emptyset = 0.$$

Let

$$\mathcal{P}^2_+ = \mathcal{P}^2 - \{(\emptyset, \emptyset)\}.$$

2.2. **Generating functions of two-partition Hodge integrals.** For $(\mu^+, \mu^-) \in \mathcal{P}^2_+$, define

$$\begin{aligned} & G_{g,\mu^+,\mu^-}(\alpha, \beta) \\ = \; & \frac{-\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}(\mu_i^+\beta + a\alpha)}{(\mu_i^+ - 1)!\alpha^{\mu_i^+-1}} \prod_{j=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_j^- -1}(\mu_j^-\alpha + a\beta)}{(\mu_j^- - 1)!\beta^{\mu_j^- -1}} \\ & \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda^\vee(\alpha)\Lambda^\vee(\beta)\Lambda^\vee(-\alpha-\beta)(\alpha\beta(\alpha+\beta))^{l(\mu^+)+l(\mu^-)-1}}{\prod_{i=1}^{l(\mu^+)}(\alpha(\alpha - \mu_i^+\psi_i)) \prod_{j=1}^{l(\mu^-)}(\beta(\beta - \mu_j^-\psi_{l(\mu^+)+j})}. \end{aligned}$$

We have the following special cases which have been studied in [21]:

$$\begin{aligned} G_{g,\mu^+,\emptyset}(\alpha, \beta) &= \frac{-\sqrt{-1}^{l(\mu^+)}}{|\operatorname{Aut}(\mu^+)|} \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}(\mu_i^+\beta + a\alpha)}{(\mu_i^+ - 1)!\alpha^{\mu_i^+-1}} \\ & \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu^+)}} \frac{\Lambda^\vee(\alpha)\Lambda^\vee(\beta)\Lambda^\vee(-\alpha-\beta)(\alpha\beta(\alpha+\beta))^{l(\mu^+)-1}}{\prod_{i=1}^{l(\mu^+)}(\alpha(\alpha - \mu_i^+\psi_i))} \\ G_{g,\emptyset,\mu^-}(\alpha, \beta) &= \frac{-\sqrt{-1}^{l(\mu^-)}}{|\operatorname{Aut}(\mu^-)|} \prod_{j=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_j^--1}(\mu_j^-\alpha + a\beta)}{(\mu_j^- - 1)!\beta^{\mu_j^--1}} \\ & \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu^-)}} \frac{\Lambda^\vee(\alpha)\Lambda^\vee(\beta)\Lambda^\vee(-\alpha-\beta)(\alpha\beta(\alpha+\beta))^{l(\mu^-)-1}}{\prod_{j=1}^{l(\mu^-)}(\beta(\beta - \mu_j^-\psi_j))} \end{aligned}$$

By a standard degree argument, one sees that $G_{g,\mu^+,\mu^-}(\alpha,\beta)$ is homogeneous of degree 0, so

$$G_{g,\mu^+,\mu^-}(\alpha,\beta) = G_{g,\mu^+,\mu^-}(1,\frac{\beta}{\alpha}).$$

Let

$$G_{g,\mu^+,\mu^-}(\tau) = G_{g,\mu^+,\mu^-}(1,\tau).$$

Introduce variables λ, $p^+ = (p_1^+, p_2^+, \ldots)$, $p^- = (p_1^-, p_2^-, \ldots)$. Given a partition μ, define

$$p_\mu^\pm = p_1^\pm \cdots p_{l(\mu)}^\pm.$$

In particular, $p_\emptyset^\pm = 1$. Define

$$\begin{aligned}
G_{\mu^+,\mu^-}(\lambda;\tau) &= \sum_{g=0}^\infty \lambda^{2g-2+l(\mu^+)+l(\mu^-)} G_{g,\mu^+,\mu^-}(\tau)\\
G(\lambda;p^+,p^-;\tau) &= \sum_{(\mu^+,\mu^-)\in\mathcal{P}_+^2} G_{\mu^+,\mu^-}(\lambda;\tau) p^+_{\mu^+} p^-_{\mu^-}\\
G^\bullet(\lambda;p^+,p^-;\tau) &= \exp(G(\lambda;p^+,p^-;\tau))\\
&= \sum_{(\mu^+,\mu^-)\in\mathcal{P}^2} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) p^+_{\mu^+} p^-_{\mu^-}\\
G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) &= \sum_{\chi\in 2\mathbb{Z},\chi\leq 2(l(\mu^+)+l(\mu^-))} \lambda^{-\chi+l(\mu^+)+l(\mu^-)} G^\bullet_{\chi,\mu^+,\mu^-}(\tau)
\end{aligned}$$

2.3. **Generating functions of representations of symmetric groups.** Let

$$q = e^{\sqrt{-1}\lambda}, \quad [m] = q^{m/2} - q^{-m/2}.$$

Define

$$\mathcal{W}_{\mu,\nu}(q) = q^{|\nu|/2}\mathcal{W}_\mu(q)\cdot s_\nu(\mathcal{E}_\mu(t)), \tag{8}$$

where

$$\mathcal{W}_\mu(q) = q^{\kappa_\mu/4} \prod_{1\leq i<j\leq l(\mu)} \frac{[\mu_i-\mu_j+j-i]}{[j-i]} \prod_{i=1}^{l(\mu)}\prod_{v=1}^{\mu_i} \frac{1}{[v-i+l(\mu)]}, \tag{9}$$

$$\mathcal{E}_\mu(t) = \prod_{j=1}^{l(\mu)} \frac{1+q^{\mu_j-j}t}{1+q^{-j}t} \cdot \left(1+\sum_{n=1}^\infty \frac{t^n}{\prod_{i=1}^n (q^i-1)}\right). \tag{10}$$

In the special case of $(\mu^+,\mu^-) = (\emptyset,\emptyset)$, we have

$$\mathcal{W}_{\emptyset,\emptyset} = 1.$$

Define

$$\begin{aligned}
R^\bullet(\lambda;p^+,p^-;\tau) &= \sum_{|\nu^\pm|=|\mu^\pm|\geq 0} \frac{\chi_{\nu^+}(C(\mu^+))}{z_{\mu^+}} \frac{\chi_{\nu^-}(C(\mu^-))}{z_{\mu^-}}\\
&\quad \cdot e^{\sqrt{-1}(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1})\lambda/2} \mathcal{W}_{\nu^+,\nu^-}(e^{\sqrt{-1}\lambda}) p^+_{\mu^+} p^-_{\mu^-}.
\end{aligned}$$

2.4. **The conjecture and the strategy.** The main purpose of this paper is to prove the following formula conjectured by the third author in [30]:

Theorem 3. *We have the following formula of two-partition Hodge integrals:*

$$G^\bullet(\lambda; p^+, p^-; \tau) = R^\bullet(\lambda; p^+, p^-; \tau). \tag{2}$$

The method in [28, 21] shows that $R^\bullet$ satisfies the cut-and-join equation (3). In [30], the third author proved Theorem 1 (initial values). So (2) follows from Theorem 2 (cut-and-join equation of $G^\bullet$). We will recall the proof of Theorem 1 in Section 2.5.

2.5. **Initial values.** For completeness, we now recall the proof of Theorem 1, which says

$$G^\bullet(\lambda; p^+, p^-; -1) = R^\bullet(\lambda; p^+, p^-; -1). \tag{4}$$

We need the skew Schur functions [22]. Recall the Schur functions are related to the Newton functions by:

$$s_\mu(x) = \sum_{|\nu|=|\mu|} \frac{\chi_\mu(\nu)}{z_\nu} p_\nu(x),$$

where $x = (x_1, x_2, \dots)$ are formal variables such that

$$p_i(x) = x_1^i + x_2^i + \cdots .$$

There are integers $c_{\mu\nu}^\eta$ such that

$$s_\mu s_\nu = \sum_\eta c_{\mu\nu}^\eta s_\eta.$$

The skew Schur functions are defined by:

$$s_{\eta/\mu} = \sum_\nu c_{\mu\nu}^\eta s_\nu.$$

Note that $p^\pm = p(x^\pm)$.

2.5.1. *The left-hand-side.* When $l(\mu^+) + l(\mu^-) > 2$,

$$G_{\mu^+,\mu^-}(\lambda; -1) = 0;$$

when $l(\mu^+) = 1$ and $l(\mu^-) = 0$,

$$\begin{aligned} & G_{\mu^+,\mu^-}(\lambda; -1) \\ = \ & -\sqrt{-1}\lambda^{-1} \sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_g}{\frac{1}{\mu_1^+}\left(\frac{1}{\mu_1^+} - \psi_1\right)} \frac{\prod_{a=1}^{\mu_1^+-1}(-\mu_1^+ + a)}{\mu_1^+ \cdot \mu_1^+!} \\ = \ & (-1)^{\mu_1^+}\sqrt{-1} \cdot \frac{1}{2\mu_1^+ \sin(\mu_1^+\lambda/2)} = \frac{(-1)^{\mu_1^+-1}}{q^{\mu_1^+/2} - q^{-\mu_1^+/2}} \cdot \frac{p_{\mu_1^+}}{\mu_1^+}; \end{aligned}$$

the case of $l(\mu^+)=0$ and $l(\mu^-)=1$ is similar; when $l(\mu^+)=l(\mu^-)=1$,

$$\begin{aligned} G_{\mu^+,\mu^-}(\lambda;-1) &= \lim_{\tau\to-1}\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,2}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-1-\tau)}{\frac{1}{\mu_1^+}\left(\frac{1}{\mu_1^+}-\psi_1\right)\cdot\frac{\tau}{\mu_1^-}\left(\frac{\tau}{\mu_1^-}-\psi_2\right)} \\ &\quad \cdot\tau(1+\tau)\cdot\frac{\prod_{a=1}^{\mu_1^+-1}\left(\mu_1^+\tau+a\right)}{\mu_1^+\cdot\mu_1^+!}\cdot\frac{\prod_{a=1}^{\mu_1^--1}\left(\frac{\mu_1^-}{\tau}+a\right)}{\mu_1^-\cdot\mu_1^-!}. \end{aligned}$$

One needs to consider the $g=0$ term and the $g>0$ terms separately. In the second case, the limit is zero while in first case, by our convention:

$$\int_{\overline{\mathcal{M}}_{0,2}}\frac{\Lambda_0^\vee(1)\Lambda_0^\vee(\tau)\Lambda_0^\vee(-1-\tau)}{\frac{1}{\mu_1^+}\left(\frac{1}{\mu_1^+}-\psi_1\right)\cdot\frac{\tau}{\mu_1^-}\left(\frac{\tau}{\mu_1^-}-\psi_2\right)}=\frac{(\mu_1^+)^2(\frac{\mu_1^-}{\tau})^2}{\mu_1^++\frac{\mu_1^-}{\tau}},$$

hence when $\mu_1^+\neq\mu_1^-$, the limit is zero, when $\mu_1^+=\mu_1^-$, the limit is:

$$\lim_{\tau\to-1}\frac{(\mu_1^+)^2(\frac{\mu_1^-}{\tau})^2}{\mu_1^++\frac{\mu_1^-}{\tau}}\cdot\tau(1+\tau)\cdot\frac{\prod_{a=1}^{\mu_1^+-1}\left(\mu_1^+\tau+a\right)}{\mu_1^+\cdot\mu_1^+!}\cdot\frac{\prod_{a=1}^{\mu_1^--1}\left(\frac{\mu_1^-}{\tau}+a\right)}{\mu_1^-\cdot\mu_1^-!}=\frac{1}{\mu_1^+}.$$

Recall that $p^\pm=p(x^\pm)$. With this notation, the initial value is:

$$\begin{aligned} & G^\bullet(\lambda;p(x^+),p(x^-);-1) \\ = & \exp\left(\sum_{n\geq1}\frac{(-1)^{n-1}}{q^{n/2}-q^{-n/2}}\frac{p_n(x^+)}{n}+\sum_{n\geq1}\frac{(-1)^{n-1}}{q^{n/2}-q^{-n/2}}\frac{p_n(x^-)}{n}+\sum_{n\geq1}\frac{p_n(x^+)p_n(x^-)}{n}\right) \\ = & \prod_{i,j=1}^{\infty}\frac{1}{(1+q^{i-1/2}x_j^+)(1+q^{i-1/2}x_j^-)}\prod_{j,k}\frac{1}{1-x_j^+x_k^-} \\ = & \sum_{\rho^+}s_{\rho^+}(-q^{1/2},-q^{3/2},\dots)s_{\rho^+}(x^+)\cdot\sum_{\rho}s_\rho(x^+)s_\rho(x^-) \\ & \cdot\sum_{\nu^-}s_{\rho^-}(-q^{1/2},-q^{3/2},\dots)s_{\rho^-}(x^-) \\ = & \sum_{\nu^\pm,\rho,\rho^\pm}s_{\rho^+}(-q^{1/2},-q^{3/2},\dots)c^{\nu^+}_{\rho^+\rho}s_{\nu^+}(x^+)\cdot c^{\nu^-}_{\rho^-\rho}s_{\nu^-}(x^-)s_{\rho^-}(-q^{1/2},-q^{3/2},\dots) \\ = & \sum_{\rho,\nu^\pm}s_{\nu^+/\rho}(-q^{1/2},-q^{3/2},\dots)s_{\nu^-/\rho}(-q^{1/2},-q^{3/2},\dots)\cdot s_{\nu^+}(x^+)s_{\nu^-}(x^-). \end{aligned}$$

2.5.2. *The right-hand side.* The following identity is proved in [30]:

$$\mathcal{W}_{\mu,\nu}(q)=(-1)^{|\mu|+|\nu|}q^{\frac{\kappa_\mu+\kappa_\nu+|\mu|+|\nu|}{2}}\sum_\rho q^{-|\rho|}s_{\mu/\rho}(1,q,\dots)s_{\nu/\rho}(1,q,\dots). \tag{11}$$

From this one gets:

$$
\begin{aligned}
& R^\bullet(\lambda; p(x^+), p(x^-); -1) \\
= & \sum_{|\mu^\pm|=|\nu^\pm|} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^-}(\mu^-)}{z_{\mu^-}} e^{-\sqrt{-1}(\kappa_{\nu^+}+\kappa_{\nu^-})\lambda/2} \mathcal{W}_{\nu^+,\nu^-}(q) p^+_{\mu^+} p^-_{\mu^-} \\
= & \sum_{\nu^\pm} s_{\nu^+}(x^+) q^{-\kappa_{\nu^+}/2} \mathcal{W}_{\nu^+,\nu^-}(q) q^{-\kappa_{\nu^-}/2} s_{\nu^-}(x^-) \\
= & \sum_{\nu^\pm} s_{\nu^+}(x^+) s_{\nu^-}(x^-) (-1)^{|\nu^+|+|\nu^-|} q^{(|\nu^+|+|\nu^-|)/2} \\
& \cdot \sum_\rho q^{-|\rho|} s_{\nu^+/\rho}(1, q, \dots) s_{\nu^-/\rho}(1, q, \dots) \\
= & \sum_{\nu^\pm} s_{\nu^+}(x^+) s_{\nu^-}(x^-) \sum_\rho s_{\nu^+/\rho}(-q^{1/2}, -q^{3/2}, \dots) s_{\nu^-/\rho}(-q^{1/2}, -q^{3/2}, \dots).
\end{aligned}
$$

The proof of Theorem 1 is complete.

3. Double Hurwitz numbers and the cut-and-join equation of $G^\bullet$

In this section, we first derive some identities of double Hurwitz numbers, such as sum formula and cut-and-join equations, which, together with initial values, characterize the double Hurwitz numbers. Then we combine these identities with (6) to obtain Theorem 2 (cut-and-join equation of $G^\bullet$).

3.1. **Double Hurwitz numbers.** Let X be a Riemann surface of genus h. Given n partitions $\eta^1, \dots, \eta^n$ of d, denote by $H^X_d(\eta^1, \dots, \eta^n)^\bullet$ and $H^X_d(\eta^1, \dots, \eta^n)^\circ$ the weighted counts of possibly disconnected and connected Hurwitz covers of type $(\eta^1, \dots, \eta^n)$ respectively. We will use the following formula for Hurwitz numbers (see e.g. [5]):

$$
H^X_d(\eta^1, \dots, \eta^n)^\bullet = \sum_{|\rho|=d} \left(\frac{\dim R_\rho}{d!} \right)^{2-2h} \prod_{i=1}^n |C_{\eta^i}| \frac{\chi_\rho(C_{\eta^i})}{\dim R_\rho}. \tag{12}
$$

It is sometimes referred to as the *Burnside formula*.

Suppose $C \to \mathbb{P}^1$ is a genus g cover which has ramification type μ^+, μ^- at two points p_0 and p_1 respectively, and ramification type (2) at r other points. By Riemann-Hurwitz formula,

$$
r = 2g - 2 + l(\mu^+) + l(\mu^-). \tag{13}
$$

Denote

$$
\begin{aligned}
H^\circ_g(\mu^+, \mu^-) &= H^{\mathbb{P}^1}_d(\mu^+, \mu^-, \eta^1, \dots, \eta^r)^\circ, \\
H^\bullet_g(\mu^+, \mu^-) &= H^{\mathbb{P}^1}_d(\mu^+, \mu^-, \eta^1, \dots, \eta^r)^\bullet,
\end{aligned}
$$

for $\eta^1 = \cdots = \eta^r = (2)$. We have by (12):

$$
H^\bullet_g(\mu^+, \mu^-) \quad = \quad \sum_{|\nu|=d} f_\nu(2)^r \frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_\nu(C_{\mu^-})}{z_{\mu^+}}, \tag{14}
$$

where r is given by (13), and

$$
f_\nu(2) = |C_{(2)}| \frac{\chi_\nu(C_{(2)})}{\dim R_\nu}.
$$

Define

$$\Phi^{\circ}_{\mu^+,\mu^-}(\lambda) = \sum_{g\geq 0} H^{\circ}_g(\mu^+,\mu^-)\frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!},$$

$$\Phi^{\bullet}_{\mu^+,\mu^-}(\lambda) = \sum_{g\geq 0} H^{\bullet}_g(\mu^+,\mu^-)\frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!},$$

$$\Phi^{\circ}(\lambda;p^+,p^-) = \sum_{\mu^+,\mu^-} \Phi^{\circ}_{\mu^+,\mu^-}(\lambda)p^+_{\mu^+}p^-_{\mu^-},$$

$$\Phi^{\bullet}(\lambda;p^+,p^-) = 1+\sum_{\mu^+,\mu^-} \Phi^{\bullet}_{\mu^+,\mu^-}(\lambda)p^+_{\mu^+}p^-_{\mu^-}.$$

The usual relationship between connected and disconnected Hurwitz numbers is:

$$\Phi^{\circ}(\lambda;p^+,p^-) = \log \Phi^{\bullet}(\lambda;p^+,p^-). \tag{15}$$

By (14) one easily gets:

$$\Phi^{\bullet}(\lambda;p^+,p^-) = 1+\sum_{d\geq 1}\sum_{|\mu^{\pm}|=d}\sum_{|\nu|=d}\frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}}\frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}}e^{f_\nu(2)\lambda}p^+_{\mu^+}p^-_{\mu^-}. \tag{16}$$

Equivalently,

$$\Phi^{\bullet}_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d}\frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}}\frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}}e^{f_\nu(2)\lambda}. \tag{17}$$

We also have

$$\Phi^{\bullet}_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d}(-1)^{l(\mu^+)+l(\mu^-)}\frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}}\frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}}e^{-f_\nu(2)\lambda}. \tag{18}$$

3.2. **Sum formula and initial values.**

Proposition 3.1. *We have*

$$\Phi^{\bullet}_{\mu^1,\mu^3}(\lambda_1+\lambda_2) = \sum_{\mu^2}\Phi^{\bullet}_{\mu^1,\mu^2}(\lambda_1)\cdot z_{\mu^2}\cdot\Phi^{\bullet}_{\mu^2,\mu^3}(\lambda_2), \tag{19}$$

$$\Phi^{\bullet}_{\mu^1,\mu^3}(0) = \frac{1}{z_{\mu^1}}\delta_{\mu^1,\mu^3}. \tag{20}$$

Proof. By the orthogonality relation for characters of S_d:

$$\sum_{\mu}\frac{\chi_{\nu^1}(C_\mu)\chi_{\nu^2}(C_\mu)}{z_\mu} = \delta_{\nu^1,\nu^2} \tag{21}$$

we have

$$\begin{aligned}
&\sum_{\mu^2} \Phi^\bullet_{\mu^1,\mu^2}(\lambda_1) \cdot z_{\mu^2} \cdot \Phi^\bullet_{\mu^2,\mu^3}(\lambda_2) \\
= &\sum_{\mu^2}\sum_{\nu^1} \frac{\chi_{\nu^1}(C_{\mu^1})}{z_{\mu^1}} \frac{\chi_{\nu^1}(C_{\mu^2})}{z_{\mu^2}} e^{f_{\nu^1}(2)\lambda_1} \cdot z_{\mu^2} \cdot \sum_{\nu^2} \frac{\chi_{\nu^2}(C_{\mu^2})}{z_{\mu^2}} \frac{\chi_{\nu^2}(C_{\mu^3})}{z_{\mu^3}} e^{f_{\nu^2}(2)\lambda_2} \\
= &\sum_{\nu^1}\sum_{\nu^2} \frac{\chi_{\nu^1}(C_{\mu^1})}{z_{\mu^1}} \frac{\chi_{\nu^2}(C_{\mu^3})}{z_{\mu^3}} e^{f_{\nu^1}(2)\lambda_1 + f_{\nu^2}(2)\lambda_2} \cdot \sum_{\mu^2} \frac{\chi_{\nu^1}(C_{\mu^2})\chi_{\nu^2}(C_{\mu^2})}{z_{\mu^2}} \\
= &\sum_{\nu^1}\sum_{\nu^2} \frac{\chi_{\nu^1}(C_{\mu^1})}{z_{\mu^1}} \frac{\chi_{\nu^2}(C_{\mu^3})}{z_{\mu^3}} e^{f_{\nu^1}(2)\lambda_1 + f_{\nu^2}(2)\lambda_2} \delta_{\nu^1,\nu^2} \\
= &\sum_{\nu} \frac{\chi_{\nu}(C_{\mu^1})}{z_{\mu^1}} \frac{\chi_{\nu}(C_{\mu^3})}{z_{\mu^3}} e^{f_{\nu}(2)(\lambda_1+\lambda_2)} \\
= &\Phi^\bullet_{\mu^1,\mu^3}(\lambda_1+\lambda_2).
\end{aligned}$$

Similarly, by the orthogonality relation:

$$\sum_{|\nu|=d} \chi_\nu(C_{\mu^1}) \cdot \chi_\nu(C_{\mu^2}) = z_{\mu^1}\delta_{\mu^1,\mu^2}. \tag{22}$$

we have

$$\Phi^\bullet_{\mu^1,\mu^2}(0) = \sum_{|\nu|=d} \frac{\chi_\nu(C_{\mu^1})}{z_{\mu^1}} \cdot \frac{\chi_\nu(C_{\mu^2})}{z_{\mu^2}} = \frac{1}{z_{\mu^1}}\delta_{\mu^1,\mu^2}.$$

□

Equation (19) is a sum formula for double Hurwitz numbers, and Equation (20) gives the initial values for double Hurwitz numbers.

Corollary 3.2. *Denote by* $\Phi^\bullet(\lambda)_d$ *the matrix* $(\Phi^\bullet_{\mu,\nu}(\lambda))_{|\mu|=|\nu|=d}$. *Then* $\Phi^\bullet(\lambda)_d$ *is invertible, and*

$$Z_d^{-1}\Phi^\bullet(-\lambda)_d^{-1} = \Phi^\bullet(\lambda)_d Z_d. \tag{23}$$

where $Z_d = (z_\mu \delta_{\mu,\nu})_{|\mu|=|\nu|=d}$.

Proof. In (19) we take $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$, then by (20) we have

$$Z_d^{-1} = \Phi^\bullet(0)_d = \Phi^\bullet(\lambda)_d Z_d \Phi^\bullet(-\lambda)_d.$$

Taking determinant on both sides one sees that $\Phi^\bullet(\lambda)_d$ is invertible, and (23) is a straightforward consequence. □

3.3. **Cut-and-join equation for double Hurwitz numbers.** Recall for any partition ν of d, one has

$$f_\nu(2) \cdot \sum_\mu \frac{\chi_\nu(C_\mu)}{z_\mu} p_\mu = \frac{1}{2}\sum_{i,j}\left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right)\sum_\eta \frac{\chi_\nu(C_\eta)}{z_\eta} p_\eta.$$

See e.g. [28, 21]. From this one easily proves the following results.

Proposition 3.3. *We have the following equations:*

$$\frac{\partial \Phi^\bullet_h}{\partial \lambda} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp^{\pm}_{i+j}\frac{\partial^2 \Phi^\bullet_h}{\partial p^{\pm}_i \partial p^{\pm}_j} + (i+j)p^{\pm}_i p^{\pm}_j \frac{\partial \Phi^\bullet_h}{\partial p^{\pm}_{i+j}}\right), \tag{24}$$

$$\frac{\partial \Phi^\circ_h}{\partial \lambda} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp^{\pm}_{i+j}\frac{\partial^2 \Phi^\circ_h}{\partial p^{\pm}_i \partial p^{\pm}_j} + ijp^{\pm}_{i+j}\frac{\partial \Phi^\circ_h}{\partial p^{\pm}_i}\frac{\partial \Phi^\circ_h}{\partial p^{\pm}_j} + (i+j)p^{\pm}_i p^{\pm}_j \frac{\partial \Phi^\circ_h}{\partial p^{\pm}_{i+j}}\right). \tag{25}$$

The new feature for the double Hurwitz numbers is that there are two choices to do the cut-and-join, on the $+$ side or on the $-$ side. One can rewrite (24) as sequences of systems of ODEs as follows. For each partition μ^- of d, one gets a system of ODEs for $\{\Phi^\bullet_{\mu^+,\mu^-}(\lambda) : |\mu^+| = d\}$, hence they are determined by $\{\Phi^\bullet_{\mu^+,\mu^-}(0) : |\mu^+| = d\}$. One can also reverse the roles of μ^+ and μ^-. There are matrices CJ_d such that the cut-and-join equations in degree d can be written as

$$\frac{d}{d\lambda}\Phi^\bullet_d = CJ_d\cdot\Phi^\bullet_d = \Phi^\bullet_d\cdot CJ^t_d. \tag{26}$$

Example 3.4. When $d = 2$, the cut-and-join equation becomes

$$\begin{aligned}\frac{d}{d\lambda}\begin{pmatrix}\Phi^\bullet_{(2),(2)} & \Phi^\bullet_{(2),(1^2)}\\ \Phi^\bullet_{(1^2),(2)} & \Phi^\bullet_{(1^2),(1^2)}\end{pmatrix} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Phi^\bullet_{(2),(2)} & \Phi^\bullet_{(2),(1^2)}\\ \Phi^\bullet_{(1^2),(2)} & \Phi^\bullet_{(1^2),(1^2)}\end{pmatrix}\\ &= \begin{pmatrix}\Phi^\bullet_{(2),(2)} & \Phi^\bullet_{(2),(1^2)}\\ \Phi^\bullet_{(1^2),(2)} & \Phi^\bullet_{(1^2),(1^2)}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\end{aligned}$$

The initial values are:

$$\begin{pmatrix}\Phi^\bullet_{(2),(2)} & \Phi^\bullet_{(2),(1^2)}\\ \Phi^\bullet_{(1^2),(2)} & \Phi^\bullet_{(1^2),(1^2)}\end{pmatrix}(0) = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$$

Hence we have the following solution:

$$\begin{pmatrix}\Phi^\bullet_{(2),(2)} & \Phi^\bullet_{(2),(1^2)}\\ \Phi^\bullet_{(1^2),(2)} & \Phi^\bullet_{(1^2),(1^2)}\end{pmatrix}(\lambda) = \begin{pmatrix}\frac{1}{2}\cosh\lambda & \frac{1}{2}\sinh\lambda\\ \frac{1}{2}\sinh\lambda & \frac{1}{2}\cosh\lambda\end{pmatrix}$$

This is compatible with (18).

3.4. **Cut-and-join equation for two-partition Hodge integrals.** For each $(\mu^+,\mu^-)\in\mathcal{P}^2_+$, we will define a generating function

$$K^\bullet_{\mu^+,\mu^-}(\lambda)$$

of relative Gromov-Witten invariants. In Section 7, we will derive the following identity by relative virtual localization:

$$K^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(-\sqrt{-1}\tau\lambda)z_{\nu^+}G^\bullet_{\nu^+,\nu^-}(\lambda;\tau)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}(\frac{-\sqrt{-1}}{\tau}\lambda). \tag{6}$$

In matrix form, one has for $d^+, d^-\geq 0$,

$$K^\bullet(\lambda)_{d^+,d^-} = \Phi^\bullet(-\sqrt{-1}\tau\lambda)_{d^+}Z_{d^+}G^\bullet(\lambda;\tau)_{d^+,d^-}Z_{d^-}\Phi^\bullet(\frac{-\sqrt{-1}}{\tau}\lambda)_{d^-}.$$

Hence by (23) we have

$$G^\bullet(\lambda;\tau)_{d^+,d^-} = \Phi^\bullet(\sqrt{-1}\tau\lambda)_{d^+}Z_{d^+}K^\bullet(\lambda)_{d^+,d^-}Z_{d^-}\Phi^\bullet(\frac{\sqrt{-1}}{\tau}\lambda)_{d^-}.$$

Taking derivative in τ on both sides, one then gets:

$$\frac{\partial}{\partial\tau}G^\bullet(\lambda;\tau)_{d^+,d^-} = \sqrt{-1}\lambda\left(CJ_{d^+}\cdot G^\bullet(\lambda;\tau)_{d^+,d^-} - \frac{1}{\tau^2}G^\bullet(\lambda;\tau)_{d^+,d^-}\cdot CJ^t_{d^-}\right).$$

This completes the proof of the cut-and-join equation for $G^\bullet$ and hence the proof of the formula (2) of two-partition Hodge integrals.

Corollary 3.5. *We have*

$$K^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{\eta^\pm}\frac{\chi_{\eta^+}(C_{\mu^+})}{z_{\mu^+}}\cdot\mathcal{W}_{\eta^+,\eta^-}(e^{\sqrt{-1}\lambda})\cdot\frac{\chi_{\eta^-}(C_{\mu^-})}{z_{\mu^-}}.$$

Proof. By (6), (2) and (17) we have

$$\begin{aligned}
& K^\bullet_{\mu^+,\mu^-}(\lambda)\\
= & \sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(-\sqrt{-1}\tau\lambda)z_{\nu^+}G^\bullet_{\nu^+,\nu^-}(\lambda;\tau)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}(\frac{-\sqrt{-1}}{\tau}\lambda)\\
= & \sum_{\nu^\pm,\rho^\pm,\eta^\pm} e^{-\sqrt{-1}f_{\rho^+}(2)\tau\lambda}\frac{\chi_{\rho^+}(C_{\mu^+})}{z_{\mu^+}}\frac{\chi_{\rho^+}(C_{\nu^+})}{z_{\nu^+}}\cdot z_{\nu^+}\\
& \cdot\frac{\chi_{\eta^+}(C_{\nu^+})}{z_{\nu^+}}e^{\sqrt{-1}\kappa_{\eta^+}\tau\lambda/2}\mathcal{W}_{\eta^+,\eta^-}(e^{\sqrt{-1}\lambda})e^{\sqrt{-1}\kappa_{\eta^-}\tau^{-1}\lambda/2}\frac{\chi_{\eta^-}(C_{\nu^-})}{z_{\nu^-}}\\
& \cdot z_{\nu^-}\cdot\frac{\chi_{\rho^-}(C_{\nu^-})}{z_{\nu^-}}\frac{\chi_{\rho^-}(C_{\mu^-})}{z_{\mu^-}}e^{-\sqrt{-1}f_{\rho^-}(2)\tau^{-1}\lambda}\\
= & \sum_{\eta^\pm}\frac{\chi_{\eta^+}(C_{\mu^+})}{z_{\mu^+}}\cdot\mathcal{W}_{\eta^+,\eta^-}(e^{\sqrt{-1}\lambda})\cdot\frac{\chi_{\eta^-}(C_{\mu^-})}{z_{\mu^-}}.
\end{aligned}$$

In the last equality we have used (21). □

4. Relative stable morphisms and relative virtual localization

In this section, we will give a brief review of the moduli spaces of algebraic relative stable morphisms [16, 17] and virtual localization on such spaces [9, 11].

4.1. **Relative stable morphisms.** The definitions given in this section are based on J. Li's works on relative stable morphisms [16, 17], with minor modifications.

Let Y be a smooth projective variety. Let $D^1,\ldots,D^k$ be disjoint smooth divisors in Y. For $\alpha = 1,\ldots,k$, define

$$\Delta(D^\alpha) = \mathbb{P}(\mathcal{O}_{D^\alpha}\oplus\mathcal{N}_{D^\alpha/Y})\to D^\alpha,$$

where $\mathcal{N}_{D/Y}$ denotes the normal sheaf of a subvariety D in Y. The projective line bundle $\Delta(D^\alpha)\to D^\alpha$ has two distinct sections

$$D^\alpha_0 = \mathbb{P}(\mathcal{O}_{D^\alpha}\oplus 0),\quad D^\alpha_\infty = \mathbb{P}(0\oplus\mathcal{N}_{D^\alpha/Y}).$$

We have

$$\mathcal{N}_{D^\alpha_0/\Delta(D^\alpha)}\cong\mathcal{N}^{-1}_{D^\alpha/Y},\quad \mathcal{N}_{D^\alpha_\infty/\Delta(D^\alpha)}\cong\mathcal{N}_{D^\alpha/Y}.$$

Let

$$\Delta(D^\alpha)(m) = \Delta(D^\alpha)_1\cup\Delta(D^\alpha)_2\cup\cdots\cup\Delta(D^\alpha)_m,$$

where $\Delta(D^\alpha)_i\cong\Delta(D^\alpha)$ for $i=1,\ldots,m$. Let $D^\alpha_{i,0}$ and $D^\alpha_{i,\infty}$ be the two distinct sections of $\Delta(D^\alpha)_i$ which correspond to D^α_0 and D^α_∞, respectively. Then $\Delta(D^\alpha)(m)$

is obtained by identifying $D^\alpha_{i,\infty}$ with $D^\alpha_{i+1,0}$ for $i = 1, \cdots, m-1$ under the canonical isomorphisms

$$D^\alpha_{i,\infty} \cong D^\alpha \cong D^\alpha_{i+1,0}.$$

Define

$$D^\alpha_{(0)} = D^\alpha_{1,0}, \quad D^\alpha_{(i)} = \Delta(D^\alpha)_i \cap \Delta(D^\alpha)_{i+1}, \quad D^\alpha_{(m)} = D^\alpha_{m,\infty},$$

where $i = 1, \ldots, m-1$. The $\mathbb{C}^*$ action on $\mathcal{O}_{D^\alpha}$ induces a $\mathbb{C}^*$ action on $\Delta(D^\alpha)$ such that $\Delta(D^\alpha) \to D^\alpha$ is $\mathbb{C}^*$ equivariant, where $\mathbb{C}^*$ acts on D^α trivially. The two distinct sections $D^\alpha_0, D^\alpha_\infty$ are fixed under this $\mathbb{C}^*$ action. So there is a $(\mathbb{C}^*)^m$ action on $\Delta(D^\alpha)(m)$ fixing $D^\alpha_{(0)}, \ldots, D^\alpha_{(m)}$, such that $\Delta(D^\alpha)(m) \to D^\alpha$ is $(\mathbb{C}^*)^m$ equivariant, where $(\mathbb{C}^*)^m$ acts on D^α trivially.

The variety

$$Y[m^1, \ldots, m^k] = Y \cup \bigcup_{\alpha=1}^{k} \Delta(D^\alpha)(m^\alpha)$$

with normal crossing singularities is obtained by identifying $D^\alpha \subset Y$ with $D^\alpha_{(0)} \subset \Delta(D^\alpha)$ under the canonical isomorphism. There is a morphism

$$\pi[m^1, \ldots, m^k] : Y[m^1, \ldots, m^k] \to Y$$

which contracts $\Delta(D^\alpha)(m^\alpha)$ to D^α. The $(\mathbb{C}^*)^{m^\alpha}$ action on $\Delta(D^\alpha)(m^\alpha)$ gives a $(\mathbb{C}^*)^{m^1+\cdots+m^k}$ on $Y[m^1, \ldots, m^k]$ such that $\pi[m^1, \ldots, m^k]$ is $(\mathbb{C}^*)^{m^1+\cdots+m^k}$ equivariant with respect to the trivial action on Y.

With the above notation, we are now ready to define relative stable morphisms for $(Y; D^1, \ldots, D^k)$.

Definition 4.1. *Let $\beta \in H_2(Y, \mathbb{Z})$ be a nonzero homology class such that*

$$d^\alpha = \int_\beta c_1(\mathcal{O}(D^\alpha)) \geq 0.$$

Let μ^α be a partition of d^α. Define

$$\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$$

to be the moduli space of morphisms

$$f : (C, \{x^1_i\}_{i=1}^{l(\mu^1)}, \ldots, \{x^k_i\}_{i=1}^{l(\mu^k)}) \to Y[m^1, \ldots, m^k]$$

such that

(1) *$(C, \{x^1_i\}_{i=1}^{l(\mu^1)}, \ldots, \{x^k_i\}_{i=1}^{l(\mu^k)})$ is a connected prestable curve of arithmetic genus g with $\sum_{\alpha=1}^k l(\mu^\alpha)$ marked points.*
(2) *$(\pi[m^1, \ldots, m^k] \circ f)_*[C] = \beta \in H_2(Y; \mathbb{Z})$.*
(3)
$$f^{-1}(D^\alpha_{(m^\alpha)}) = \sum_{i=1}^{l(\mu^\alpha)} \mu^\alpha_i x^\alpha_i$$
as Cartier divisors. In particular, if $d^\alpha = 0$, then $f^{-1}(D^\alpha_{(m^\alpha)})$ is empty.
(4) *The preimage of $D^\alpha_{(l)}$ consists of nodes of C, where $0 \leq l \leq m^\alpha - 1$. If $f(y) \in D^\alpha_{(l)}$ and C_1 and C_2 are two irreducible components of C which intersect at y, then $f|_{C_1}$ and $f|_{C_2}$ have the same contact order to $D^\alpha_{(l)}$ at y.*
(5) *The automorphism group of f is finite.*

Two morphisms described above are isomorphic if they differ by an isomorphism of the domain and an element in $(\mathbb{C}^*)^{m^1+\cdots+m^k}$ *acting on the target. In particular, this defines the automorphism group in the stability condition (5) above.*

Remark 4.2. *In* [16, 17], *the number of divisors* $k=1$, *but the construction and proofs in* [16, 17] *show that*

$$\overline{\mathcal{M}}_{g,0}(Y;D^1,\ldots,D^k \mid \beta;\mu^1,\ldots,\mu^k)$$

is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$\int_\beta c_1(TY)+(1-g)(\dim Y-3)+\sum_{\alpha=1}^k (l(\mu^\alpha)-|\mu^\alpha|),$$

where TY *is the tangent bundle of* Y*.*

Definition 4.3. *We define the moduli space* $\overline{\mathcal{M}}^\bullet_\chi(Y;D^1,\ldots,D^k \mid \beta;\mu^1,\cdots,\mu^k)$ *similarly, with (1) replaced by the following (1)*$^\bullet$*, and one additional condition (6):*

(1)$^\bullet$ $(C,\{x_i^1\}_{i=1}^{l(\mu^1)},\ldots,\{x_i^k\}_{i=1}^{l(\mu^k)})$ *is a possibly disconnected prestable curve with* $\sum_{\alpha=1}^k l(\mu^\alpha)$ *marked points. Let* $C_1,\ldots,C_n$ *be the connected components of* C*, and let* g_i *be the arithmetic genus of* C_i*. Then*

$$\sum_{i=1}^n (2-2g_i)=\chi.$$

(6) *Let* $\beta_i=\tilde{f}_*[C_i]$*, where* C_i *is a connected component of* C*. Then* $\beta_i\neq 0$*, and*

$$\int_{\beta_i} c_1(\mathcal{O}(D^\alpha))\geq 0$$

for $\alpha=1,\ldots,k$*.*

The moduli space

$$\overline{\mathcal{M}}^\bullet_\chi(Y;D^1,\ldots,D^k \mid \beta;\mu^1,\cdots,\mu^k)$$

is a finite quotient of a disjoint union of products of the moduli spaces defined in Definition 4.1. By [16, 17], it is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$\int_\beta c_1(TY)+\frac{\chi}{2}(\dim Y-3)+\sum_{\alpha=1}^k (l(\mu^\alpha)-|\mu^\alpha|).$$

4.2. **Tangent and obstruction spaces.** This section is based on [17, Section 5.1]. We first introduce some notation. If $m^\alpha>0$, define line bundles L_l^α on $D^\alpha_{(l)}\subset Y[m^1,\ldots,m^k]$ by

$$L_l^\alpha=\begin{cases} N_{D^\alpha_{(0)}/Y}\otimes N_{D^\alpha_{(0)}/\Delta(D^\alpha)_1} & l=0\\ N_{D^\alpha_{(l)}/\Delta(D^\alpha)_l}\otimes N_{D^\alpha_{(l)}/\Delta(D^\alpha)_{l+1}} & 1\leq l\leq m^\alpha-1\end{cases}$$

Note that L_l^α is a trivial line bundle on $D^\alpha_{(l)}$.

The tangent space T^1 and the obstruction space T^2 of

$$\overline{\mathcal{M}}_{g,0}(Y;D^1,\ldots,D^k \mid \beta;\mu^1,\ldots,\mu^k)$$

at the moduli point

$$\left[f:(C,\{x_i^1\}_{i=1}^{l(\mu^1)},\dots,\{x_i^k\}_{i=1}^{l(\mu^k)})\to Y[m^1,\dots,m^k]\,\right]$$

are given by the following two exact sequences:

$$\begin{aligned}0\quad&\to \mathrm{Ext}^0(\Omega_C(R),\mathcal{O}_C)\to H^0(\mathbf{D}^\bullet)\to T^1\\ &\to \mathrm{Ext}^1(\Omega_C(R),\mathcal{O}_C)\to H^1(\mathbf{D}^\bullet)\to T^2\quad\to 0\end{aligned}\tag{27}$$

(28)

$$\begin{aligned}0\quad\to\quad & H^0\left(C,f^*\left(\Omega_{Y[m^1,\dots,m^k]}(\sum_{\alpha=1}^k\log D^\alpha_{(m^\alpha)})\right)^\vee\right)\to H^0(\mathbf{D}^\bullet)\to\bigoplus_{\alpha=1}^k\bigoplus_{l=0}^{m^\alpha-1}H^0_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\\ \to\quad & H^1\left(C,f^*\left(\Omega_{Y[m^1,\dots,m^k]}(\sum_{\alpha=1}^k\log D^\alpha_{(m^\alpha)})\right)^\vee\right)\to H^1(\mathbf{D}^\bullet)\to\bigoplus_{\alpha=1}^k\bigoplus_{l=0}^{m^\alpha-1}H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\to 0\end{aligned}$$

where

$$R=\sum_{\alpha=1}^k\sum_{i=1}^{l(\mu^\alpha)}x_i^\alpha,$$

$$H^0_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\cong\bigoplus_{q\in f^{-1}(D^\alpha_{(l)})}T_q(f^{-1}(\Delta(D^\alpha)_l))\otimes T_q^*(f^{-1}(\Delta(D^\alpha)_l))\cong\mathbb{C}^{\oplus n_l^\alpha},\tag{29}$$

$$H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\cong H^0(D^\alpha_{(l)},L_l^\alpha)^{\oplus n_l^\alpha}\Big/H^0(D^\alpha_{(l)},L_l^\alpha),\tag{30}$$

and n_l^α is the number of nodes over D_l^α. In (30),

$$H^0(D^\alpha_{(l)},L_l^\alpha)\to H^0(D^\alpha_{(l)},L_l^\alpha)^{\oplus n_l^\alpha}$$

is the diagonal embedding.

We refer the reader to [17] for the definitions of $H^i(\mathrm{D}^\bullet)$ and the maps between terms in (27), (28). Here we only explain the part relevant to virtual localization calculations. The vector space

$$B_1=\mathrm{Ext}^0(\Omega_C(R),\mathcal{O}_C)$$

is the space of the infinitesimal automorphisms of the domain curve (C,R), and

$$B_4=\mathrm{Ext}^1(\Omega_C(R),\mathcal{O}_C)$$

is the space of the infinitesimal deformations of (C,R). Let $\hat{C}$ be the normalization of C, $\hat{R}\subset\hat{C}$ be the pull back of R, and $R'\subset\hat{C}$ be the divisor corresponding to nodes in C. From the local to global spectral sequence, we have an exact sequence

$$0\to B_{4,0}\to B_4\to B_{4,1}\to 0,$$

where

$$B_{4,0}=H^1(C,\mathcal{E}xt^0_{\mathcal{O}_C}(\Omega_C(R),\mathcal{O}_C))=H^1(C,\Omega_C(R)^\vee)$$

is the space of infinitesimal deformations of the smooth pointed curve $(\hat{C},\hat{R}+R')$, and

$$B_{4,1}=H^0(C,\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega_C(R),\mathcal{O}_C))\cong\bigoplus_{q\in\mathrm{Sing}(C)}T_{q'}\hat{C}\otimes T_{q''}\hat{C}$$

corresponds to smoothing of nodes of the domain curve. Here $\mathrm{Sing}(C)$ is the set of nodes of C, and $q', q'' \in \hat{C}$ are the two preimages of q under the normalization map $\hat{C} \to C$. The tangent line of smoothing of the node q is canonically identified with $T_{q'}\hat{C} \otimes T_{q''}\hat{C}$.

The complex vector space

$$B_2 = H^0\left(C, f^*\left(\Omega_{Y[m^1,\ldots,m^k]}(\sum_{\alpha=1}^k \log D^\alpha_{(m^\alpha)})\right)^\vee\right)$$

is the space of infinitesimal deformations of the map f with fixed domain and target, and

$$B_5 = H^1\left(C, f^*\left(\Omega_{Y[m^1,\ldots,m^k]}(\sum_{\alpha=1}^k \log D^\alpha_{(m^\alpha)})\right)^\vee\right)$$

is the obstruction space to deforming f with fixed domain and target.

Finally, let

$$B_3 = \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H^0_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet}), \quad B_6 = \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet}).$$

The complex vector space $H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})$ correponds to obstruction to smoothing the nodes in $f^{-1}(D^\alpha_{(l)})$. More explicitly, let

$$f^{-1}(D^\alpha_{(l)}) = \{q_1, \ldots, q_n\},$$

and let ν_i be the contact order of f to $D^\alpha_{(l)}$ at q_i (of either of the two branches of f near q_i). Then $B_4 \to H^1(\mathbf{D}^\bullet)$ in (27) induces a map

$$\begin{array}{rcl} \bigoplus_{i=1}^n T_{q_{i,1}}\hat{C} \otimes T_{q_{i,2}}\hat{C} & \to & H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet}) \cong H^0(D^\alpha_{(l)}, L^\alpha_l)^{\oplus n} \Big/ H^0(D^\alpha_{(l)}, L^\alpha_l) \\ (s_1, \ldots, s_n) & \mapsto & [(s_1^{\nu_1}, \ldots, s_n^{\nu_n})] \end{array}$$

where we use isomorphisms

$$H^0(D^\alpha_{(l)}, L^\alpha_l) \cong (L^\alpha_l)_{f(q_i)} \cong \left(T_{q_i'}\hat{C} \otimes T_{q_i''}\hat{C}\right)^{\otimes \nu_i}.$$

The first isomorphism follows from the triviality of the line bundle $L^\alpha_l \to D^\alpha_{(l)}$. We see that the obstruction vanishes iff the smoothing of the nodes $q_1, \ldots, q_n$ is compatible with the smoothing the target along the divisor $D^\alpha_{(l)}$, which is parametrized by the complex line $H^0(D^\alpha_{(l)}, L^\alpha_l)$.

4.3. **Relative virtual localization.** In this section, we assume that a torus $T = (\mathbb{C}^*)^r$ acts on Y, and $D^1, \ldots, D^k$ are T-invariant divisors.

Under our assumption, $\mathcal{N}_{D^\alpha/Y} \to D^\alpha$ is T-equivariant, and the T-action extends to $\Delta(D^\alpha)$. So T acts on $Y[m^1, \ldots, m^k]$, and acts on $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$ by moving the image.

The T fixed points set $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)^T$ is a disjoint union of

$$\{\mathcal{F}_\Gamma \mid \Gamma \in G_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)\},$$

where each $\Gamma \in G_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$ corresponds to a connected component, or a union of connected components, $\mathcal{F}_\Gamma$ of $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)^T$. Let

$$[f : (C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \ldots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \in \mathcal{F}_\Gamma \subset \overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k),$$

for some $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$. The T-action on $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$ induces T-actions on the exact sequences (27), (28) which define T^1 and T^2. Let $T^{i,f}$ and $T^{i,m}$ denote the fixing part and the moving part of T^i under the T-action, respectively, where $i = 1, 2$. Then

$$T^{1,f} - T^{2,f}$$

defines a perfect obstruction theory on $\mathcal{F}_\Gamma$, and

$$T^{1,m} - T^{2,m}$$

defines the virtual normal bundle $N^{\mathrm{vir}}_{\mathcal{F}_\Gamma}$ of $\mathcal{F}_\Gamma$ in $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$. More explicitly, let B_i^m denote the moving part of B_i under T-action, where $i = 1, \ldots, 6$. Then $B_3^m = 0$. Note that there are subtleties due to the $(\mathbb{C}^*)^{m^1 + \cdots + m^k}$ action on the target $Y[m^1, \ldots, m^k]$. We have

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})} = \frac{e_T(T^{2,m})}{e_T(T^{1,m})} = \frac{e_T(B_1^m) e_T(B_5^m) e_T(B_6^m)}{e_T(B_2^m) e_T(B_4^m)}$$

In [9], T. Graber and R. Pandharipande proved a localization formula for the virtual fundamental class in the general context of $\mathbb{C}^*$-equivariant perfect obstruction theory. In [11], T. Graber and R. Vakil showed that moduli spaces of relative stable morphisms satisfy the technical assumptions required in the general formalism in [9], and derived relative virtual localization under the assumption that the divisor is fixed pointwisely under the $\mathbb{C}^*$ action [11, Theorem 3.6]. In our context, the localization formula proved in [9] reads:

(31)

$$[\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)]^{\mathrm{vir}}_T = \sum_{\Gamma \in G_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)} (i_{\mathcal{F}_\Gamma})_* \left(\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})} \right)$$

where

$$i_{\mathcal{F}_\Gamma} : \mathcal{F}_\Gamma \to \overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$$

is the inclusion, $e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})$ is the T-equivariant Euler class of the virtual normal bundle $N^{\mathrm{vir}}_{\mathcal{F}_\Gamma} = T^{1,m} - T^{2,m}$ over $\mathcal{F}_\Gamma$,

$$[\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)]^{\mathrm{vir}}_T \in A^T_*(\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k); \mathbb{Q})$$

is the T-equivariant virtual fundamental class defined by the T-equivariant perfect obstruction theory $T^1 - T^2$ on $\overline{\mathcal{M}}_{g,0}(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \ldots, \mu^k)$, and

$$[\mathcal{F}_\Gamma]^{\mathrm{vir}}_T \in A^T_*(\mathcal{F}_\Gamma; \mathbb{Q})$$

is the T-equivariant virtual fundamental class defined by the perfect obstruction theory $T^{1,f} - T^{2,f}$ on $\mathcal{F}_\Gamma$.

Similarly, we have

(32)

$$[\overline{\mathcal{M}}^\bullet_\chi(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \cdots, \mu^k)]^{\mathrm{vir}} = \sum_{\Gamma \in G^\bullet_\chi(Y; D^1, \ldots, D^k \mid \beta; \mu^1, \cdots, \mu^k)} (i_{\mathcal{F}_\Gamma})_* \left(\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}}{e_T(N^{\mathrm{vir}}_{\mathcal{F}_\Gamma})} \right)$$

5. Double Hurwitz numbers as relative Gromov-Witten invariants

In this section, we study double Hurwitz numbers by relative Gromov-Witten theory.

5.1. **Relative morphisms to $\mathbb{P}^1$.**

Let $[Z_0, Z_1]$ be the homogeneous coordinates of $\mathbb{P}^1$. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by

$$t \cdot [Z_0, Z_1] = [tZ_0, Z_1]$$

for $t \in \mathbb{C}^*$, $[Z_0, Z_1] \in \mathbb{P}^1$. Let

$$s^+ = [0,1], \quad s^- = [1,0]$$

be the two fixed points of this $\mathbb{C}^*$-action.

Let μ^+ and μ^- be two partitions of $d > 0$. Let $[\mathbb{P}^1] \in H_2(\mathbb{P}^1;\mathbb{Z})$ be the fundamental class. Define

$$\begin{aligned}\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) &= \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, s^+, s^-; d[\mathbb{P}^1], \mu^+, \mu^-),\\ \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-) &= \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, s^+, s^-; d[\mathbb{P}^1], \mu^+, \mu^-).\end{aligned}$$

The virtual dimension of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$ is

$$2g - 2 + l(\mu^+) + l(\mu^-),$$

and the virtual dimension of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)$ is

$$-\chi + l(\mu^+) + l(\mu^-).$$

We extend the $\mathbb{C}^*$ action on $\mathbb{P}^1$ to $\mathbb{P}^1[m^+, m^-]$ by trivial action on $\Delta^\pm[m^\pm]$, which is a chain of $m^\pm$ copies of $\mathbb{P}^1$. This induces $\mathbb{C}^*$-actions on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$ and $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)$. Define the moduli spaces of unparametrized relative stable maps to the triple $(\mathbb{P}^1, s^+, s^-)$ to be

$$\begin{aligned}\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^* &= \Big(\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) \setminus \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*}\Big)\Big/ \mathbb{C}^*,\\ \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^* &= \Big(\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-) \setminus \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*}\Big)\Big/ \mathbb{C}^*.\end{aligned}$$

Then $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$2g - 2 + l(\mu^+) + l(\mu^-) - 1,$$

and $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$-\chi + l(\mu^+) + l(\mu^-) - 1.$$

5.2. **Target ψ classes.**

In the notation in Section 4.1, we have $\Delta^\pm \cong \mathbb{P}^1$, $\Delta^\pm(m)$ is a chain of m copies of $\mathbb{P}^1$, and $D^\pm_{(l)}$ is a point, for $l = 0, \ldots, m^\pm$. Let $\mathbb{L}^\pm$ and be the line bundle on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*$ whose fiber at

$$\big[f : (C, x_1, \ldots, x_{l(\mu^+)}, y_1, \ldots, y_{l(\mu^-)}) \to \mathbb{P}^1[m^+, m^-]\big] \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$$

is the cotangent line

$$T^*_{D^\pm_{(m^\pm)}}\left(\mathbb{P}^1[m^+, m^-]\right)$$

of $\mathbb{P}^1[m^+, m^-]$ at the smooth point $D^{\pm}_{(m^\pm)}$. We define $\mathbb{L}^+$ and $\mathbb{L}^-$ on $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*$, similarly. Define the *target ψ classes*

$$\psi^0 = c_1(\mathbb{L}^+), \quad \psi^\infty = c_1(\mathbb{L}^-).$$

The following integral of ψ^0 arises in the localization calculations in [21]:

$$\int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^0)^{2g-2+l(\mu^+)+l(\mu^-)-1}.$$

In Section 5.8, we will relate such integrals of target ψ classes to double Hurwitz numbers (Proposition 5.4, 5.5).

5.3. **Double Hurwitz numbers.** Let μ^+, μ^- be two partitions of $d > 0$. There are branch morphisms

$$\begin{aligned}
\mathrm{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) &\to \mathrm{Sym}^{2g-2+l(\mu^+)+l(\mu^-)}\,\mathbb{P}^1 \cong \mathbb{P}^{2g-2+l(\mu^+)+l(\mu^-)} \\
\mathrm{Br} : \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-) &\to \mathrm{Sym}^{-\chi+l(\mu^+)+l(\mu^-)}\,\mathbb{P}^1 \cong \mathbb{P}^{-\chi+l(\mu^+)+l(\mu^-)}
\end{aligned}$$

The double Hurwitz numbers for connected covers of $\mathbb{P}^1$ can be defined by

$$H^\circ_g(\mu^+, \mu^-) = \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu^+,\mu^-)]^{\mathrm{vir}}} \mathrm{Br}^*(H^{2g-2+l(\mu^+)+l(\mu^-)})$$

where $H \in H^2(\mathbb{P}^{2g-2+l(\mu^+)+l(\mu^-)}; \mathbb{Z})$ is the hyperplane class. The double Hurwitz numbers for possibly disconnected covers of $\mathbb{P}^1$ can be defined by

$$H^\bullet_{\chi,\mu^+,\mu^-} = \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)]^{\mathrm{vir}}} \mathrm{Br}^*(H^{-\chi+l(\mu^+)+l(\mu^-)}).$$

We have

$$H^\bullet_{2-2g,\mu^+,\mu^-} = H^\bullet_g(\mu^+, \mu^-).$$

Recall that $H^\circ_g(\mu^+, \mu^-), H^\bullet_g(\mu^+, \mu^-)$ are defined combinatorially in Section 3.

We define generating functions of double Hurwitz numbers as in Section 3:

$$\begin{aligned}
\Phi^\circ_{\mu^+,\mu^-}(\lambda) &= \sum_{g=0}^\infty \frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!} H^\circ_g(\mu^+, \mu^-) \\
\Phi^\bullet_{\mu^+,\mu^-}(\lambda) &= \sum_{\chi\in 2\mathbb{Z}, \frac{\chi}{2}\leq \min\{l(\mu^+), l(\mu^-)\}} \frac{\lambda^{-\chi+l(\mu^+)+l(\mu^-)}}{(-\chi+l(\mu^+)+l(\mu^-))!} H^\bullet_{\chi,\mu^+,\mu^-} \\
\Phi^\circ(\lambda; p^+, p^-) &= \sum_{\mu^+,\mu^-} \Phi^\circ_{\mu^+,\mu^-}(\lambda) p^+_{\mu^+} p^-_{\mu^-} \\
\Phi^\bullet(\lambda; p^+, p^-) &= 1 + \sum_{\mu^+,\mu^-} \Phi^\bullet_{\mu^+,\mu^-}(\lambda) p^+_{\mu^+} p^-_{\mu^-}
\end{aligned}$$

Then

$$\Phi^\bullet(\lambda; p^+, p^-) = \exp(\Phi^\circ(\lambda; p^+, p^-)).$$

Note that

$$\Phi^\bullet_{\mu^+,\mu^-}(0) = \frac{\delta_{\mu^+,\mu^-}}{z_{\mu^+}}, \tag{33}$$

where $z_\nu = \nu_1 \cdots \nu_{l(\nu)} |\operatorname{Aut}(\nu)|$, so

$$\Phi^\bullet(0, p^+, p^-) = 1 + \sum_\mu \frac{p^+_\mu p^-_\mu}{z_\mu}.$$

5.4. **Gluing formula.** Let k^+, k^- be positive integers such that

$$k^+ + k^- = -\chi + l(\mu^+) + l(\mu^-).$$

By gluing formula of algebraic relative Gromov-Witten invariants [17, Corollary 3.16], we have

$$\begin{aligned} & \int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)]^{\mathrm{vir}}} \mathrm{Br}^*(H^{-\chi+l(\mu^+)+l(\mu^-)}) \\ = & \sum_{-\chi^\pm + l(\mu^\pm)+l(\nu)=k^\pm} \int_{[\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu)]^{\mathrm{vir}}} \mathrm{Br}^*(H^{k^+}) \\ & \cdot \frac{a_\nu}{|\operatorname{Aut}(\nu)|} \int_{[\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu,\mu^-)]^{\mathrm{vir}}} \mathrm{Br}^*(H^{k^-}) \end{aligned}$$

where

$$a_\nu = \nu_1 \cdots \nu_{l(\nu)}.$$

Therefore, we have the following gluing formula for double Hurwitz numbers:

Proposition 5.1 (gluing formula)**.** *Let k^+, k^- be positive integers such that*

$$k^+ + k^- = -\chi + l(\mu^+) + l(\mu^-).$$

Then

$$H^\bullet_{\chi,\mu^+,\mu^-} = \sum_{-\chi^\pm + l(\mu^\pm)+l(\nu)=k^\pm} H^\bullet_{\chi^+,\mu^+,\nu} z_\nu H^\bullet_{\chi^-,\nu,\mu^-} \tag{34}$$

Recall that $z_\nu = a_\nu |\operatorname{Aut}(\nu)|$.

Let $d = |\mu^+| = |\mu^-|$. It is straightforward to check that Proposition 5.1 implies the *sum formula*

$$\sum_{|\nu|=d} \Phi^\bullet_{\mu^+,\nu}(\lambda_1) z_\nu \Phi^\bullet_{\nu,\mu^-}(\lambda_2) = \Phi^\bullet_{\mu^+,\mu^-}(\lambda_1+\lambda_2) \tag{35}$$

which was derived in Section 3.2 from the combinatoric definition.

The cut-and-join equations (26) for double Hurwitz numbers are special cases $k_+ = 1$, $k_- = 1$ of Proposition 5.1. More precisely, differentiate (35) with repect to λ_1, and then set $\lambda_1 = 0$. We obtain a cut-and-join equation:

$$\frac{d}{d\lambda} \Phi^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d} H^\bullet_{l(\mu^+)+l(\nu)-1,\mu^+,\nu} z_\nu \Phi^\bullet_{\nu,\mu^-}(\lambda). \tag{36}$$

Differentiate (35) with repect to λ_2, and then set $\lambda_2 = 0$. We obtain another cut-and-join equation:

$$\frac{d}{d\lambda} \Phi^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d} \Phi^\bullet_{\mu^+,\nu}(\lambda) z_\nu H^\bullet_{l(\nu)+l(\mu^-)-1,\nu,\mu^-}. \tag{37}$$

Define the *cut-and-join coefficients*

$$(CJ)_{\mu\nu} = H^\bullet_{l(\mu)+l(\nu)-1,\mu,\nu} z_\nu.$$

They are the entries of the matrix CJ_d in Section 3.3. The cut-and-join equations can be written as

$$\frac{d}{d\lambda}\Phi^{\bullet}_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d}(CJ)_{\mu^+\nu}\Phi^{\bullet}_{\nu,\mu^-}(\lambda) = \sum_{|\nu|=d}\Phi^{\bullet}_{\mu^+,\nu}(\lambda)(CJ)_{\mu^-\nu}, \tag{38}$$

which is equivalent to (26) in Section 3.3:

$$\frac{d}{d\lambda}\Phi^{\bullet}_d = CJ_d \cdot \Phi^{\bullet}_d = \Phi^{\bullet}_d \cdot CJ^t_d.$$

Remark 5.2. *The cut-and-join equation of Hurwitz numbers $H^{\circ}_g(\mu), H^{\bullet}_g(\mu)$ was first proved using combinatorics by Goulden, Jackson and Vainstein* [8] *and later proved using gluing formula of symplectic relative Gromov-Witten invariants by Li-Zhao-Zheng* [19] *and Ionel-Parker* [13].

5.5. **Localization.** In the spirit of [21, Section 7], we lift

$$H^{-\chi+l(\mu^+)+l(\mu^-)} \in H^{2(-\chi+l(\mu^+)+l(\mu^-))}(\mathbb{P}^{-\chi+l(\mu^+)+l(\mu^-)};\mathbb{Z})$$

to

$$\prod_{k=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_k u) \in H^{2(-\chi+l(\mu^+)+l(\mu^-))}_{\mathbb{C}^*}(\mathbb{P}^{-\chi+l(\mu^+)+l(\mu^-)};\mathbb{Z}),$$

where $w_k \in \mathbb{Z}$, and compute

$$H^{\bullet}_{\chi,\mu^+,\mu^-} = \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1,\mu^+,\mu^-)]^{\mathrm{vir}}}\mathrm{Br}^*\left(\prod_{k=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_k u)\right)$$

by virtual localization.

5.6. **Torus fixed points and admissible triples.** Given a morphism

$$f:(C,\{x^1_i\}_{i=1}^{l(\mu^1)},\ldots,\{x^k_i\}_{i=1}^{l(\mu^k)}) \to \mathbb{P}^1[m^+,m^-]$$

which represents a point in $\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1,\mu^+,\mu^-)^{\mathbb{C}^*}$, let

$$\tilde{f} = \pi[m^+,m^-]\circ f : C \to \mathbb{P}^1,$$

and let $C^{\pm} = \tilde{f}^{-1}(s^{\pm})$. Then

$$C = C^+ \cup L \cup C^-,$$

where L is a disjoint union of projective lines. Let

$$\begin{aligned} f^{\pm} = f|_{C^{\pm}} : C^{\pm} &\to \Delta^{\pm}(m^{\pm}),\\ f^0 = f|_L : L &\to \mathbb{P}^1. \end{aligned}$$

Then f^0 is a morphism of degree

$$d = |\mu^+| = |\mu^-|$$

fully ramified over s^+ and s^-. The degrees of f^0 restricted to connected components of L determine a partition ν of d.

Let $C^+_1,\ldots,C^+_k$ be the connected components of C^+, and let g_i be the arithmetic genus of C^+_i. (We define $g_i = 0$ if C^+_i is a point.) Define

$$\chi^+ = \sum_{i=1}^{k}(2-2g_i),$$

and define χ^- similarly. We have

$$-\chi^+ + 2l(\nu) - \chi^- = -\chi.$$

Note that $\chi^\pm \leq 2\min\{l(\mu^\pm), l(\nu)\}$. So

$$-\chi^+ + l(\mu^+) + l(\nu) \geq 0,$$

and the equality holds if and only if $m^+ = 0$. In this case, we have $\nu = \mu^+$, $\chi^+ = 2l(\mu^+)$, and $\chi^- = \chi$. Similarly,

$$-\chi^- + l(\nu) + l(\mu^-) \geq 0,$$

and the equality holds if and only if $m^- = 0$. In this case, we have $\nu = \mu^-$, $\chi^- = 2l(\mu^-)$, and $\chi^+ = \chi$. There are three cases:
Case 1: $m^- = 0$. Then f^- is a constant map, $\chi^+ = \chi$, $\nu = \mu^-$, and f^+ represents a point in

$$\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*.$$

Case 2: $m^+ = 0$. Then f^+ is a constant map, $\chi^- = \chi$, $\nu = \mu^+$, and f^- represents a point in

$$\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*.$$

Case 3: $m^+, m^- > 0$. Up to an element of $\mathrm{Aut}(\nu)$, f^+ represents a point in

$$\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1, \mu^+, \nu)//\mathbb{C}^*,$$

and f^- represents an element of

$$\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1, \nu, \mu^-)//\mathbb{C}^*.$$

Definition 5.3. *We say a triple (χ^+, ν, χ^-) is* admissible *if*

- $\chi^+, \chi^- \in 2\mathbb{Z}$.
- *ν is a partition of d.*
- $\chi^\pm \leq 2\min\{l(\mu^\pm), l(\nu)\}$.
- $-\chi^+ + 2l(\nu) - \chi^- = -\chi$.

Let $G^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)$ denote the set of all admissible triples.

We define

$$\begin{aligned} \overline{\mathcal{M}}_{\chi, \mu^-, 2l(\mu^-)} &= \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*, \\ \overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi} &= \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)//\mathbb{C}^*, \end{aligned}$$

and define

$$\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-} = \left(\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1, \mu^+, \nu)//\mathbb{C}^*\right) \times \left(\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1, \nu, \mu^-)//\mathbb{C}^*\right).$$

if $(\chi^+, \nu, \chi^-) \in G^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)$, and

$$-\chi^+ + l(\mu^+) + l(\nu) > 0, \quad -\chi^- + l(\nu) + l(\mu^-) > 0.$$

For every $(\chi^+, \nu, \chi^-) \in G^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)$, there is a morphism

$$i_{\chi^+, \nu, \chi^-} : \overline{\mathcal{M}}_{\chi^+, \nu, \chi^-} \to \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-),$$

whose image $\mathcal{F}_{\chi^+, \nu, \chi^-}$ is a union of connected components of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*}$. The morphism i_{χ^+, ν, χ^-} induces an isomorphism

$$\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-}/A_{\chi^+, \nu, \chi^-} \cong \mathcal{F}_{\chi^+, \nu, \chi^-},$$

where

$$A_{\chi,\mu^-,2l(\mu^-)} = \prod_{i=1}^{l(\mu^-)} \mathbb{Z}_{\mu_i^-}, \quad A_{2l(\mu^+),\mu^+,\chi} = \prod_{i=1}^{l(\mu^+)} \mathbb{Z}_{\mu_i^+},$$

and for $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$, we have

$$1 \to \prod_{i=1}^{l(\nu)} \mathbb{Z}_{\nu_i} \to A_{\chi^+,\nu,\chi^-} \to \mathrm{Aut}(\nu) \to 1.$$

Recall that $a_\nu = \nu_1 \cdots \nu_{l(\nu)}$, and $z_\nu = a_\nu |\mathrm{Aut}(\nu)|$. We have

$$|A_{\chi,\mu^-,2l(\mu^-)}| = a_{\mu^-}, \quad |A_{2l(\mu^+),\mu^+,\chi}| = a_{\mu^+},$$

and

$$|A_{\chi^+,\nu,\chi^-}| = z_\nu$$

if $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$.

The fixed points set $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)^{\mathbb{C}^*}$ is a disjoint union of

$$\{\mathcal{F}_{\chi^+,\nu,\chi^-} \mid (\chi^+,\nu,\chi^-) \in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)\}$$

5.7. **Contribution from each admissible triple.** Let $(\chi^+,\nu,\chi^-) \in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)$. We have

$$\begin{aligned}\mathrm{Br}(\mathcal{F}_{\chi^+,\nu,\chi^-}) &= (-\chi^+ + l(\mu^+) + l(\nu))s^+ + (-\chi^- + l(\nu) + l(\mu^-))s^- \\ &\in \mathrm{Sym}^{-\chi+l(\mu^+)+l(\mu^-)}\,\mathbb{P}^1 = \mathbb{P}^{-\chi+l(\mu^+)+l(\mu^-)},\end{aligned}$$

so

$$\begin{aligned}&i^*_{\chi^+,\nu,\chi^-}\,\mathrm{Br}^*\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)} (H - w_l)\right) \\ = &\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)} (-\chi^+ + l(\mu^+) + l(\nu) - w_l)\right) u^{-\chi+l(\mu^+)+l(\mu^-)}.\end{aligned}$$

Let $N^{\mathrm{vir}}_{\chi^+,\nu,\chi^-}$ on $\overline{\mathcal{M}}_{\chi^+,\nu,\chi^-}$ be the pull-back of the virtual normal bundle of $\mathcal{F}_{\chi^+,\nu,\chi^-}$ in $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)$. Calculations similar to those in [21, Appendix A] show that

$$\begin{aligned}\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\chi,\mu^-,2l(\mu^-)})} &= \frac{a_{\mu^-}}{u - \psi^\infty}, \\ \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{2l(\mu^+),\mu^+,\chi})} &= \frac{a_{\mu^+}}{-u - \psi^0},\end{aligned}$$

and for $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$, we have

$$\frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\chi^+,\nu,\chi^-})} = \frac{a_\nu}{u - \psi^\infty_+}\frac{a_\nu}{-u - \psi^0_-},$$

where ψ^∞_+, ψ^0_- are the target ψ classes on

$$\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu), \quad \overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu,\mu^-),$$

respectively.

Let $w=(w_1,\ldots,w_l)$. Then

$$
\begin{aligned}
& I_{\chi,\mu^-,2l(\mu^-)}(w)\\
=\ & \frac{1}{a_{\mu^-}}\int_{[\overline{\mathcal{M}}_{\chi,\mu^-,2l(\mu^-)}]^{\mathrm{vir}}} \frac{i^*_{\chi,\mu^-,2l(\mu^-)}\mathrm{Br}^*\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_l u)\right)}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\chi,\mu^-,2l(\mu^-)})}\\
=\ & \left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(-\chi+l(\mu^+)+l(\mu^-)-w_l)\right)\int_{[\overline{\mathcal{M}}_{\chi,\mu^-,2l(\mu^-)}]^{\mathrm{vir}}}\frac{u^{-\chi+l(\mu^+)+l(\mu^-)}}{u-\psi^\infty}\\
=\ & \left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(-\chi+l(\mu^+)+l(\mu^-)-w_l)\right)\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}}(\psi^\infty)^{-\chi+l(\mu^+)+l(\mu^-)-1}
\end{aligned}
$$

$$
\begin{aligned}
& I_{2l(\mu^+),\mu^+,\chi}(w)\\
=\ & \frac{1}{a_{\mu^+}}\int_{[\overline{\mathcal{M}}_{2l(\mu^+),\mu^+,\chi}]^{\mathrm{vir}}} \frac{i^*_{2l(\mu^+),\mu^+,\chi}\mathrm{Br}^*\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_l u)\right)}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{2l(\mu^+),\mu^+,\chi})}\\
=\ & \left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(-w_l)\right)\int_{[\overline{\mathcal{M}}_{2l(\mu^+),\mu^+,\chi}]^{\mathrm{vir}}}\frac{u^{-\chi+l(\mu^+)+l(\mu^-)}}{-u-\psi^0}\\
=\ & \left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}w_l\right)\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}}(\psi^0)^{-\chi+l(\mu^+)+l(\mu^-)-1}
\end{aligned}
$$

$$
\begin{aligned}
& I_{\chi^+,\nu,\chi^-}(w)\\
=\ & \frac{1}{z_\nu}\int_{[\overline{\mathcal{M}}_{\chi^+,\nu,\chi^-}]^{\mathrm{vir}}} \frac{i^*_{\chi^+,\nu,\chi^-}\mathrm{Br}^*\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_l u)\right)}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{\chi^+,\nu,\chi^-})}\\
=\ & \frac{a_\nu}{|\mathrm{Aut}(\nu)|}\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(-\chi^++l(\mu^+)+l(\nu)-w_l)\right)\int_{[\overline{\mathcal{M}}_{\chi^+,\nu,\chi^-}]^{\mathrm{vir}}}\frac{u^{-\chi+l(\mu^+)+l(\mu^-)}}{(u-\psi^\infty_+)(-u-\psi^0_-)}\\
=\ & \frac{a_\nu}{|\mathrm{Aut}(\nu)|}\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(-\chi^++l(\mu^+)+l(\nu)-w_l)\right)(-1)^{-\chi^-+l(\nu)+l(\mu^-)}\\
& \cdot\int_{[\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu)//\mathbb{C}^*]^{\mathrm{vir}}}(\psi^\infty)^{-\chi^++l(\mu^+)+l(\nu)-1}\int_{[\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}}(\psi^0)^{-\chi^-+l(\nu)+l(\mu^-)-1}
\end{aligned}
$$

5.8. **Sum over admissible triples.** We have

$$
\begin{aligned}
H^\bullet_{\chi,\mu^+,\mu^-} &= \frac{1}{|\mathrm{Aut}(\mu^+)||\mathrm{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)]^{\mathrm{vir}}}\mathrm{Br}^*\left(\prod_{l=1}^{-\chi+l(\mu^+)+l(\mu^-)}(H-w_l u)\right)\\
&= \frac{1}{|\mathrm{Aut}(\mu^+)||\mathrm{Aut}(\mu^-)|}\sum_{(\chi^+,\nu,\chi^-)\in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)} I_{\chi^+,\nu,\chi^-}(w).
\end{aligned}
$$

Let $w=(0,1,\dots,-\chi+l(\mu^+)+l(\mu^-)-1)$, we have

$$\begin{aligned} H^\bullet_{\chi,\mu^+,\mu^-} &= \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} I_{\chi,\mu^-,2l(\mu^-)}(w) \\ &= \frac{(-\chi+l(\mu^+)+l(\mu^-))!}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^\infty)^{-\chi+l(\mu^+)+l(\mu^-)-1}. \end{aligned}$$

Let $w=(1,2,\dots,-\chi+l(\mu^+)+l(\mu^-))$, we have

$$\begin{aligned} H^\bullet_{\chi,\mu^+,\mu^-} &= \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} I_{2l(\mu^+),\mu^+,\chi}(w) \\ &= \frac{(-\chi+l(\mu^+)+l(\mu^-))!}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^0)^{-\chi+l(\mu^+)+l(\mu^-)-1}. \end{aligned}$$

So we have

Proposition 5.4.

$$\begin{aligned} & \frac{H^\bullet_{\chi,\mu^+,\mu^-}}{(-\chi+l(\mu^+)+l(\mu^-))!} \\ = & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^0)^{-\chi+l(\mu^+)+l(\mu^-)-1} \\ = & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^\infty)^{-\chi+l(\mu^+)+l(\mu^-)-1} \end{aligned}$$

If we replace $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)$ by $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu^+,\mu^-)$ in Section 5.5, we get

Proposition 5.5.

$$\begin{aligned} & \frac{H^\circ_g(\mu^+,\mu^-)}{(2g-2+l(\mu^+)+l(\mu^-))!} \\ = & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^0)^{-\chi+l(\mu^+)+l(\mu^-)-1} \\ = & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu^+,\mu^-)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^\infty)^{-\chi+l(\mu^+)+l(\mu^-)-1} \end{aligned}$$

Let $w=(0,1,\dots,k-1,k+1,\dots,-\chi+l(\mu^+)+l(\mu^-))$, where

$$1\le k\le -\chi+l(\mu^+)+l(\mu^-)-1,$$

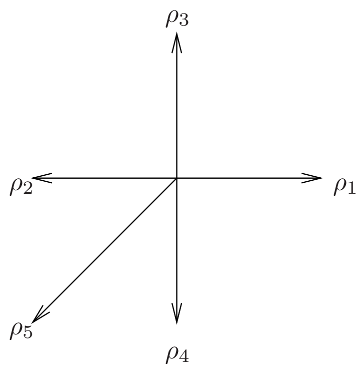

FIGURE 1. The fan of X

we have

$$\begin{aligned}
& H^\bullet_{\chi,\mu^+,\mu^-} \\
= \quad & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} \sum_{\substack{(\chi^+,\nu,\chi^-)\in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)\\ -\chi^++l(\mu^+)+l(\nu)=k}} I_{\chi^+,\nu,\chi^-}(w) \\
= \quad & \sum_{\substack{(\chi^+,\nu,\chi^-)\in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)\\ -\chi^++l(\mu^+)+l(\nu)=k}} \frac{(-\chi^++l(\mu^+)+l(\nu))!}{|\operatorname{Aut}(\mu^+)|}\int_{[\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu)]^{\mathrm{vir}}} (\psi^\infty)^{-\chi^++l(\mu^+)+l(\nu)-1} \\
& \cdot \frac{a_\nu}{|\operatorname{Aut}(\nu)|}\cdot\frac{(-\chi^-+l(\nu)+l(\mu^-))!}{|\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu,\mu^-)]^{\mathrm{vir}}}(\psi^0)^{-\chi^-+l(\nu)+l(\mu^-)-1} \\
= \quad & \sum_{\substack{(\chi^+,\nu,\chi^-)\in G^\bullet_\chi(\mathbb{P}^1,\mu^+,\mu^-)\\ -\chi^++l(\mu^+)+l(\nu)=k}} H^\bullet_{\chi^+,\mu^+,\nu} z_\nu H^\bullet_{\chi^-,\nu,\mu^-}.
\end{aligned}$$

This gives an alternative derivation of the gluing formula (34), and in particular, the cut-and-join equations (36), (37).

6. Moduli spaces and obstruction bundles

In this section, we introduce the geometric objects involved in the proof of (6), and fix notation.

6.1. **The target X.** Let X be the toric surface defined by the fan in Figure 1. Let Φ_i be the homogeneous coordinate associated to the ray ρ_i, $i=1,\ldots,5$, and set

$$\begin{aligned}
Z_{ij} &= \{(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5)\in\mathbb{C}^5 \mid \Phi_i=\Phi_j=0\},\\
Z &= Z_{12}\cup Z_{35}\cup Z_{24}\cup Z_{15}\cup Z_{34}.
\end{aligned}$$

Then

$$X = (\mathbb{C}^5\setminus Z)/(\mathbb{C}^*)^3,$$

where $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^5$ by

$$(u_1,u_2,u_3)\cdot(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) = (u_1\Phi_1, u_1u_3\Phi_2, u_2\Phi_3, u_2u_3\Phi_4, u_3^{-1}\Phi_5),$$

for $(u_1,u_2,u_3)\in(\mathbb{C}^*)^3$, $(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5)\in\mathbb{C}^5$.

$T = (\mathbb{C}^*)^2$ acts on X by

$$(t_1, t_2) \cdot [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] = [t_1\Phi_1, \Phi_2, t_2\Phi_3, \Phi_4, \Phi_5]$$

for $(t_1, t_2) \in T$, $[\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] \in X$.

Let

$$D_i = \{[\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] \in X \mid \Phi_i = 0\} \subset X$$

be the T-invariant divisor associated to the ray ρ_i. Let $\beta_i \in H_2(X;\mathbb{Z})$ be the homology class represented by D_i. We have

$$\begin{aligned} H_2(X;\mathbb{Z}) &= \left(\bigoplus_{i=1}^{5} \mathbb{Z}\beta_i\right) / \left(\mathbb{Z}(\beta_1 - \beta_2 - \beta_5) \oplus \mathbb{Z}(\beta_3 - \beta_4 - \beta_5)\right) \\ &= \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_3 \oplus \mathbb{Z}\beta_5 \end{aligned}$$

Let $\beta_i^* \in H^2(X;\mathbb{Z})$ be the Poincare dual of β_i, $i = 1, \ldots, 5$. The intersection form on

$$H^2(X;\mathbb{Z}) = \mathbb{Z}\beta_1^* \oplus \mathbb{Z}\beta_3^* \oplus \mathbb{Z}\beta_5^*$$

is given by

	β_1^*	β_3^*	β_5^*
β_1^*	0	1	0
β_3^*	1	0	0
β_5^*	0	0	-1

So

$$\beta_2^* \cdot \beta_2^* = \beta_4^* \cdot \beta_4^* = -1, \quad \beta_2^* \cdot \beta_4^* = 0.$$

Note that X is a toric blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, and D_5 is the exceptional divisor. More explicitly, we have

$$\begin{aligned} h : X &\to \mathbb{P}^1 \times \mathbb{P}^1 \\ [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] &\mapsto ([\Phi_1, \Phi_2\Phi_5], [\Phi_3, \Phi_4\Phi_5]) \end{aligned}$$

which is an isomorphism outside D_5, and $h(D_5) = \{([1,0],[1,0])\}$.

The T-invariant divisor

$$K_X = -D_1 - D_2 - D_3 - D_4 - D_5$$

is a canonical divisor of X, so

$$c_1(T_X) = 2\beta_1^* + 2\beta_3^* - \beta_5^*.$$

For $(\mu^+, \mu^-) \in \mathcal{P}_+^2$, define

$$\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-) = \overline{\mathcal{M}}_{g,0}(X; D_2, D_4 \mid |\mu^+|\beta_3 + |\mu^-|\beta_1; \mu^+, \mu^-),$$

and let $\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)$ be the subset of

$$\overline{\mathcal{M}}_\chi^\bullet(X; D_2, D_4 \mid |\mu^+|\beta_3 + |\mu^-|\beta_1; \mu^+, \mu^-)$$

which consists of morphisms

$$f : C \to X[m^+, m^-]$$

such that for each connected component C_i of C, $\tilde{f}_*[C_i] \in H_2(X;\mathbb{Z})$ is an element of

$$\{a\beta_3 + b\beta_1 \mid a, b \in \mathbb{Z}_{\geq 0}, (a,b) \neq (0,0)\}.$$

The virtual dimension of $\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$ is

$$r_{g,\mu^+,\mu^-} = g - 1 + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-),$$

and the virtual dimension of $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$ is

$$r^\bullet_{\chi,\mu^+,\mu^-} = -\frac{\chi}{2} + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-).$$

The moduli space $\overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$ plays the role of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ in the proof of Mariño-Vafa formula [21].

We have

$$D_2 \cong \mathbb{P}^1 \cong D_4, \quad \mathcal{N}_{D_2/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{N}_{D_4/X},$$

so

$$\Delta(D_2) \cong \mathbb{F}_1 \cong \Delta(D_4)$$

in the notation of Section 4.1, where $\mathbb{F}_1$ is the Hirzebruch surface

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1.$$

6.2. **The obstruction bundles.** Let

$$\pi : \mathcal{U}_{g,\mu^+,\mu^-} \to \overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$$

be the universal domain curve, and let

$$P : \mathcal{T}_{g,\mu^+,\mu^-} \to \overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$$

be the universal target. There is an evaluation map

$$F : \mathcal{U}_{g,\mu^+,\mu^-} \to \mathcal{T}_{g,\mu^+,\mu^-}$$

and a contraction map

$$\tilde{\pi} : \mathcal{T}_{g,\mu^+,\mu^-} \to X.$$

Let $\mathcal{D}_{g,\mu^+,\mu^-} \subset \mathcal{U}_{g,\mu^+,\mu^-}$ be the divisor corresponding to the $l(\mu^+)+l(\mu^-)$ marked points. Define

$$V_{g,\mu^+,\mu^-} = R^1\pi_*\left(\tilde{F}^*\mathcal{O}_X(-D_1-D_3)\otimes \mathcal{O}_{\mathcal{U}_{g,\mu^+,\mu^-}}(-\mathcal{D}_{g,\mu^+,\mu^-})\right)$$

where $\tilde{F} = \tilde{\pi}\circ F : \mathcal{U}_{g,\mu^+,\mu^-} \to X$. The fibers of V_{g,μ^+,μ^-} at

$$\left[f : (C, x_1,\ldots,x_{l(\mu^+)}, y_1,\ldots,y_{l(\mu^-)}) \to X[m^+,m^-]\right] \in \overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$$

is

$$H^1(C, \tilde{f}^*\mathcal{O}_X(-D_1-D_3)\otimes\mathcal{O}_C(-R))$$

where $\tilde{f} = \pi[m^+,m^-]\circ f$, and

$$R = x_1 + \ldots + x_{l(\mu^+)} + y_1 + \cdots + y_{l(\mu^-)}.$$

Note that

$$H^0(C, \tilde{f}^*\mathcal{O}_X(-D_1-D_3)\otimes\mathcal{O}_C(-R)) = 0,$$

and

$$\deg \tilde{f}^*\mathcal{O}_X(-D_1-D_3)\otimes\mathcal{O}_C(-R) = -|\mu^+| - |\mu^-| - l(\mu^+) - l(\mu^-),$$

so $V_{g,\mu^+,\mu^-} \to \overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$ is a vector bundle of rank

$$r_{g,\mu^+,\mu^-} = g - 1 + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-).$$

The vector bundle $V_{g,\mu^+,\mu^-} \to \overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$ plays the role of the obstruction bundle $V \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ in the proof of Mariño-Vafa formula [21, Section 4.4].

Similarly, we define a vector bundle $V^\bullet_{\chi,\mu^+,\mu^-}$ of rank

$$r^\bullet_{\chi,\mu^+,\mu^-} = -\frac{\chi}{2} + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-)$$

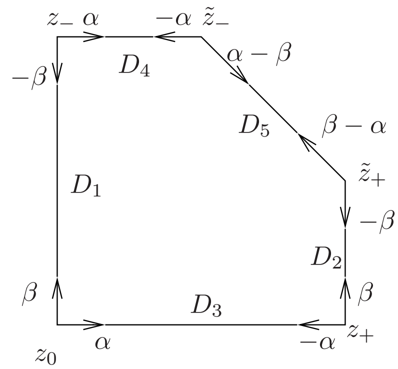

FIGURE 2. The image of $\mu_{T_{\mathbb{R}}} : X \to \mathfrak{t}_{\mathbb{R}}^*$

on $\overline{\mathcal{M}}^{\bullet}_{\chi}(X, \mu^+, \mu^-)$.

6.3. **Torus action.** Recall that $T = (\mathbb{C}^*)^2$ acts on X by

$$(t_1, t_2) \cdot [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] = [t_1\Phi_1, \Phi_2, t_2\Phi_3, \Phi_4, \Phi_5]$$

for $(t_1, t_2) \in T$, $[\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] \in X$.

Let $T_{\mathbb{R}} = U(1)^2$ be the maximal compact subgroup of T The $T_{\mathbb{R}}$-action on X determines a moment map

$$\mu_{T_{\mathbb{R}}} : X \to \mathfrak{t}_{\mathbb{R}}^*,$$

where $\mathfrak{t}_{\mathbb{R}}^* \cong \mathbb{R}^2$ is the dual of the Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of $T_{\mathbb{R}}$.

We now lift the T-action on X to the line bundle $\mathcal{O}_X(-D_1 - D_3)$ as follows. We only need to specify the representation of T on the fiber of one fixed point of the T action. The fixed points of the T action on X are

$$\begin{array}{rcl} z_0 = D_1 \cap D_3 & = & [0,1,0,1,1] \\ z_+ = D_3 \cap D_2 & = & [1,0,0,1,1] \\ z_- = D_1 \cap D_4 & = & [0,1,1,0,1] \\ \tilde{z}_+ = D_2 \cap D_5 & = & [1,0,1,1,0] \\ \tilde{z}_- = D_4 \cap D_5 & = & [1,1,1,0,0] \end{array}$$

Figure 2 shows the image of $D_1, \ldots, D_5$ and the above five fixed points under the moment map $\mu_{T_{\mathbb{R}}} : X \to \mathfrak{t}_{\mathbb{R}}^*$.

Let (w_1, w_2) denote the one dimensional representation given by

$$(t_1, t_2) \cdot z = t_1^{w_1} t_2^{w_2} z$$

for $(t_1, t_2) \in T$, $z \in \mathbb{C}$. The character ring of T is given by

$$\mathbb{Z}T \cong \mathbb{Z}[\alpha, \beta],$$

where α, β are the characters of the representations $(1, 0)$, $(0, 1)$, respectively. The representations of T on the fibers of T_X and $\mathcal{O}_X(-D_1 - D_3)$ at fixed points are

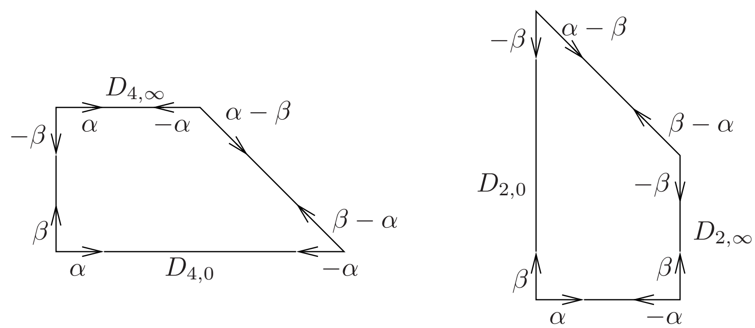

FIGURE 3. The images of $\mu^-_{T_\mathbb{R}}:\Delta(D_4)\to \mathfrak{t}^*_\mathbb{R}$ and $\mu^+_{T_\mathbb{R}}:\Delta(D_2)\to \mathfrak{t}^*_\mathbb{R}$

given by:

	T_X	$\mathcal{O}_X(-D_1-D_3)$
z_0	α,β	$-\alpha-\beta$
z_+	$-\alpha,\beta$	$-\beta$
z_-	$\alpha,-\beta$	$-\alpha$
$\tilde{z}_+$	$\beta-\alpha,-\beta$	0
$\tilde{z}_-$	$\alpha-\beta,-\alpha$	0

Note that

$$\mathfrak{t}^*_\mathbb{R}\cong \mathbb{R}\alpha\oplus\mathbb{R}\beta,$$

and the representations of T on the fibers of T_X at the fixed points can be read off from the image of the moment map as in Figure 3.

The action of T on $\Delta(D_2)$ and $\Delta(D_4)$ can be read off from Figure 3. This extends the action of T on X to $X[m^+,m^-]$. So T acts on $\overline{\mathcal{M}}_{g,0}(X,\mu^+,\mu^-)$ and $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$ by moving the image of the morphism.

The T action on $\mathcal{O}_X(-D_1-D_3)$ induces T actions on V_{g,μ^+,μ^-} and on $V^\bullet_{\chi,\mu^+,\mu^-}$.

7. PROOF OF (6)

Let X be defined as in Section 6.1. Recall that $T=(\mathbb{C}^*)^2$ acts on X. Let $D_1,\ldots,D_5$ be T-invariant divisors in X defined in Section 6.1.

Let

$$V^\bullet_{\chi,\mu^+,\mu^-}\to\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$$

be defined as in Section 6.2, with the torus action defined in Section 6.3. Define

$$\begin{aligned} K^\bullet_{\chi,\mu^+,\mu^-} &= \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)]^{\mathrm{vir}}} e(V^\bullet_{\chi,\mu^+,\mu^-}),\\ K^\bullet_{\mu^+,\mu^-}(\lambda) &= \sum_{\chi\in 2\mathbb{Z},\chi\leq 2(l(\mu^+)+l(\mu^-))}\lambda^{-\chi+l(\mu^+)+l(\mu^-)}\frac{(-1)^{|\mu^+|+|\mu^-|}}{\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}K^\bullet_{\chi,\mu^+,\mu^-}. \end{aligned}$$

In this section, we will compute

$$K^\bullet_{\chi,\mu^+,\mu^-}=\frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)]^{\mathrm{vir}}} e_T(V^\bullet_{\chi,\mu^+,\mu^-})$$

by relative virtual localization, and derive the following identity:

Proposition 7.1.

$$K^\bullet_{\mu^+,\mu^-}(\lambda)=\sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(-\sqrt{-1}\tau\lambda)z_{\nu^+}G^\bullet_{\nu^+,\nu^-}(\lambda;\tau)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}(\frac{-\sqrt{-1}}{\tau}\lambda). \tag{6}$$

7.1. **Torus fixed points.** Given a morphism

$$(C,\{x_i\}_{i=1}^{l(\mu^+)},\{y_j\}_{j=1}^{l(\mu^-)})\to X[m^+,m^-]$$

which represents a point in $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)^T$, let

$$\tilde{f}=\pi[m^+,m^-]\circ f:C\to X.$$

Then

$$\tilde{f}(C)\subset D_1\cup D_2\cup D_3\cup D_4\cup D_5,$$
$$\tilde{f}(x_i)\in\{z_+,\tilde{z}_+\},\quad \tilde{f}(y_j)\in\{z_-,\tilde{z}_-\}.$$

See Figure 3 in Section 6.3 for the configuration of the T-invariant divisors $D_1,\ldots,D_5$ and the T fixed points $z_0,z_+,z_-,\tilde{z}_+,\tilde{z}_-$.

If

$$\tilde{f}_*(C)=n_1D_1+n_2D_2+n_3D_3+n_4D_4+n_5D_5$$

as divisors, then

$$\tilde{f}_*[C]=(n_1+n_2)\beta_1+(n_3+n_4)\beta_3+(n_5-n_2-n_4)\beta_5$$

as homology classes.

Let

$$J=\{(n_1,n_2,n_3,n_4,n_5)\in\mathbb{Z}^5\mid n_i\geq 0, n_1+n_2=|\mu^-|, n_3+n_4=|\mu^+|, n_5=n_2+n_4\}.$$

Given $\hat{n}=(n_1,n_2,n_3,n_4,n_5)\in J$, let

$$\mathcal{M}_{\hat{n}}\subset\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)^T$$

be the subset which corresponds to

$$\tilde{f}_*(C)=n_1D_1+n_2D_2+n_3D_3+n_4D_4+n_5D_5.$$

Then $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)^T$ is a disjoint union of

$$\{\mathcal{M}_{\hat{n}}:\hat{n}\in J\}.$$

We have the following vanishing lemma:

Lemma 7.2. *Let $\hat{n}\in J$, and let $i_{\hat{n}}:\mathcal{M}_{\hat{n}}\to\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)^T$ be the inclusion. Then*

$$i^*_{\hat{n}}e_T(V^\bullet_{\chi,\mu^+,\mu^-})=0$$

unless $\hat{n}=(|\mu^-|,0,|\mu^+|,0,0)$.

Proof. We use the notation in Section 6.2. Let $L=\mathcal{O}_X(-D_1-D_3)$. We have the following short exact sequence of sheaves on $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$:

$$0\to\tilde{F}^*L(-\mathcal{D}^\bullet_{\chi,\mu^+,\mu^-})\to\tilde{F}^*L\to(\tilde{F}^*L)_{\mathcal{D}^\bullet_{\chi,\mu^+,\mu^-}}\to 0. \tag{39}$$

Let

$$s_i:\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)\to\mathcal{U}^\bullet_{\chi,\mu^+,\mu^-}$$

be the section corresponds to the i-th marked point,

$$\mathrm{ev}_i=\tilde{F}\circ s_i:\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)\to X$$

be evaluation at the i-th marked point. Then (39) gives the following long exact sequence:

$$\begin{aligned} 0 \quad \to \quad & R^0\pi_*\tilde{F}^*L(-D^\bullet_{\chi,\mu^+,\mu^-}) \to R^0\tilde{F}^*L \to \bigoplus_{i=1}^{l(\mu^+)+l(\mu^-)} \mathrm{ev}_i^*L \\ \to \quad & R^1\pi_*\tilde{F}^*L(-D^\bullet_{\chi,\mu^+,\mu^-}) \to R^1\tilde{F}^*L \to 0. \end{aligned}$$

We have $R^0\tilde{F}^*L = 0$, so

$$\tilde{V}^\bullet_{\chi,\mu^+,\mu^-} = R^1\tilde{F}^*L$$

is a vector bundle over $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$. We have the following short exact sequence of vector bundles over $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$:

$$0 \to \bigoplus_{i=1}^{l(\mu^+)+l(\mu^-)} \mathrm{ev}_i^*L \to V^\bullet_{\chi,\mu^+,\mu^-} \to \tilde{V}^\bullet_{\chi,\mu^+,\mu^-} \to 0. \tag{40}$$

The restriction of the above exact sequence to $\mathcal{M}_{\hat{n}}$ is

$$0 \to L_{z_-}^{\oplus l(\sigma^1)} \oplus L_{\tilde{z}_-}^{\oplus l(\sigma^2)} L_{z_+}^{\oplus l(\sigma^3)} \oplus L_{\tilde{z}_-}^{\oplus l(\sigma^4)} \to i_{\hat{n}}^* V^\bullet_{\chi,\mu^+,\mu^-} \to i_{\hat{n}}^* \tilde{V}^\bullet_{\chi,\mu^+,\mu^-} \to 0,$$

where $\sigma^1,\ldots,\sigma^4$ are partitions determined by

$$\{\mu_j^- : \tilde{f}(y_j) \in z_-\}, \quad \{\mu_j^- : \tilde{f}(y_j) \in \tilde{z}_-\}, \quad \{\mu_i^+ : \tilde{f}(x_i) \in z_+\}, \quad \{\mu_i^+ : \tilde{f}(x_i) \in \tilde{z}_+\}, \tag{41}$$

respectively. Note that $\sigma^1,\sigma^2,\sigma^3,\sigma^4$ are constant on each connected components of $\mathcal{M}_{\hat{n}}$, and

$$\sigma^1 \cup \sigma^2 = \mu^-, \quad \sigma^3 \cup \sigma^4 = \mu^+.$$

We have seen in Section 6.3 that

$$e_T(L_{z_+}) = -\beta, \quad e_T(L_{z_-}) = -\alpha, \quad e_T(L_{\tilde{z}^+}) = e_T(L_{\tilde{z}^-}) = 0,$$

so

$$i^* e_T(V^\bullet_{\chi,\mu^+,\mu^-}) = 0$$

unless

$$(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = (\mu^-,\emptyset,\mu^+,\emptyset). \tag{42}$$

Let $\hat{n} = (n_1,n_2,n_3,n_4,n_5) \in J$, $n_5 \neq 0$. Let $\mathcal{M}_{\hat{n}}(k)$ be the subset of $\mathcal{M}_{\hat{n}}$ which consists of points

$$\big[f : C \to X[m^+,m^-]\big] \in \mathcal{M}_{\hat{n}}$$

such that (42) is true, and

$$f^{-1}(D_5 - \{\tilde{z}^+,\tilde{z}^-\})$$

has k connected components, where $1 \leq k \leq n_5$. Each $\mathcal{M}_{\hat{n}}(k)$ is a union of connected components of $\mathcal{M}_{\hat{n}}$.

We claim that

$$e_T(V^\bullet_{\chi,\mu^+,\mu^-})|_{\mathcal{M}_{\hat{n}}(k)} = 0$$

for all $\hat{n} = (n_1,n_2,n_3,n_4,n_5) \in J$, $n_5 \neq 0$, $k = 1,\ldots,n_5$. This will complete the proof.

Let

$$\big[f : C \to X[m^+,m^-]\big] \in \mathcal{M}_{\hat{n}}(k).$$

Then $C = C_1 \cup C_2$, where C_1 is the closure of $f^{-1}(D_5 - \{\tilde{z}^+,\tilde{z}^-\})$, which is a disjoint union of k projective lines, and C_2 is the union of other irreducible components of

C. By (42), the ramification divisor $R \subset C_2$, and C_1 and C_2 intersect at $2k$ nodes. We have

$$\begin{aligned} 0 \quad \to \quad & H^0(C, \tilde{f}^* L(-R)) \to H^0(C_1, \tilde{f}^* L|_{C_1}) \oplus H^0(C_2, \tilde{f}^* L(-R)|_{C_2}) \to L_{\tilde{z}^+}^{\oplus k} \oplus L_{\tilde{z}^-}^{\oplus k} \\ \to \quad & H^1(C, \tilde{f}^* L(-R)) \to H^1(C_1, \tilde{f}^* L|_{C_1}) \oplus H^1(C_2, \tilde{f}^* L(-R)|_{C_2}) \to 0 \end{aligned}$$

where

$$H^0(C, \tilde{f}^* L(-R)) = 0 = H^0(C_2, \tilde{f}^* L(-R)|_{C_2}).$$

The restriction of L to D_5 is (equivariantly) trivial, so

$$H^0(C_1, \tilde{f}^* L|C_1) \cong L_{\tilde{z}^-}^{\oplus k}, \quad H^1(C_1, \tilde{f}^* L|C_1) = 0.$$

We have

$$0 \to L_{\tilde{z}^+}^{\oplus k} \to H^1(C, \tilde{f}^* L) \to H^1(C_2, \tilde{f}^* L) \to 0,$$

so

$$V^\bullet_{\chi,\mu^+,\mu^-}|_{\mathcal{M}_{\hat{n}}(k)} = L^k_{\tilde{z}^+} \oplus V',$$

and

$$e_T(V^\bullet_{\chi,\mu^+,\mu^-})|_{\mathcal{M}_{\hat{n}}(k)} = 0.$$

□

Lemma 7.2 tells us that $\mathcal{M}_{\hat{n}}$ does not contribute to the localization calculation of

$$K^\bullet_{\chi,\mu^+,\mu^-} = \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)]^{\mathrm{vir}}} e_T(V^\bullet_{\chi,\mu^+,\mu^-})$$

if $\hat{n} \neq (|\mu^-|, 0, |\mu^+|, 0, 0)$.

7.2. **Admissible labels.** From now on, we only consider

$$\hat{\mathcal{M}} = \mathcal{M}_{(|\mu^-|,0,|\mu^+|,0,0)} \subset \overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)^T.$$

Given a morphism

$$(C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \ldots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \to X[m^+, m^-]$$

which represents a point in $\hat{\mathcal{M}}$, let

$$C^0 = \tilde{f}^{-1}(z_0), \quad C^\pm = \tilde{f}^{-1}(z_\pm),$$

where z_0, z_+, z_- are defined as in Section 6.3. Then

$$C = C^+ \cup L^+ \cup C^0 \cup L^- \cup C^-,$$

where L^+, L^- are unions of projective lines, $f|_{L^+} : L^+ \to D_3$ is a degree $d^+ = |\mu^+|$ cover fully ramified over z_0 and z_+, and $f|_{L^-} : L^- \to D_1$ is a degree $d^- = |\mu^-|$ cover fully ramified over z_0 and z_-.

Define

$$\mathbb{P}^\pm(m^\pm) = \pi[m^+, m^-]^{-1}(z_\pm).$$

Let

$$\begin{aligned} f^\pm = f|_{C^\pm} : C^\pm \quad &\to \quad \mathbb{P}^\pm(m^\pm), \\ \tilde{f}^+ = f|_{L^+} : L^+ \quad &\to \quad D_3, \\ \tilde{f}^- = f|_{L^-} : L^- \quad &\to \quad D_1. \end{aligned}$$

The degrees of $\tilde{f}^\pm$ restricted to irreducible components of $L^\pm$ determine a partition $\nu^\pm$ of $d^\pm$.

Let $C^0_1,\dots,C^0_k$ be the connected components of C^0, and let g_i be the arithmetic genus of C^0_i. (We define $g_i=0$ if C^0_i is a point.) Define

$$\chi^0=\sum_{i=1}^k(2-2g_i).$$

We define χ^+,χ^- similarly. Then

$$-\chi^+ +2l(\nu^+)-\chi^0+2l(\nu^-)-\chi^-=-\chi.$$

Note that $\chi^\pm\leq 2\min\{l(\mu^\pm),l(\nu^\pm)\}$. So

$$-\chi^+ +l(\nu^+)+l(\mu^+)\geq 0,$$

and the equality holds if and only if $m^+=0$. In this case, we have $\nu^+=\mu^+$, $\chi^+=2l(\mu^+)$. Similarly,

$$-\chi^- +l(\nu^-)+l(\mu^-)\geq 0,$$

and the equality holds if and only if $m^-=0$. In this case, we have $\nu^-=\mu^-$, $\chi^-=2l(\mu^-)$. There are four cases:
Case 1: $m^+=m^-=0$. Then f^+,f^- are constant maps, and $\nu^\pm=\mu^\pm$.
Case 2: $m^+>0,m^-=0$. Then f^- is a constant map, $\nu^-=\mu^-$, and f^+ represents a point in

$$\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu^+)//\mathbb{C}^*$$

up to an element in $\mathrm{Aut}(\nu^+)$.
Case 3: $m^+=0,m^->0$. Then f^+ is a constant map, $\nu^+=\mu^+$, and f^- represents a point in

$$\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu^-,\mu^-)//\mathbb{C}^*$$

up to an element in $\mathrm{Aut}(\nu^-)$.
Case 4: $m^+,m^->0$. Then f^+ represents a point in

$$\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\mu^+,\nu^+)//\mathbb{C}^*$$

up to an element of $\mathrm{Aut}(\nu^+)$, and f^- represents an point in

$$\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\nu^-,\mu^-)//\mathbb{C}^*$$

up to an element in $\mathrm{Aut}(\nu^-)$.

Definition 7.3. *An* admissible label *is a 5-uple* $(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-)$ *such that*

- $\chi^+,\chi^0,\chi^-\in 2\mathbb{Z}$.
- $\nu^\pm$ *is a partition of* $d^\pm$.
- $\chi^0\leq 2\min\{l(\nu^+),l(\nu^-)\}$, $\chi^\pm\leq 2\min\{l(\mu^\pm),l(\nu^\pm)\}$.
- $-\chi^+ +2l(\nu^+)-\chi^0+2l(\nu^-)-\chi^-=-\chi$.

Let $G^\bullet_\chi(X,\mu^+,\mu^-)$ *denote the set of all admissible labels.*

For a nonnegative integer g and a positive integer h, let $\overline{\mathcal{M}}_{g,h}$ be the moduli space of stable curves of genus g with h marked points. $\overline{\mathcal{M}}_{g,h}$ is empty for $(g,h)=(0,1),(0,2)$, but we will assume that $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ exist and satisfy

$$\int_{\overline{\mathcal{M}}_{0,1}}\frac{1}{1-d\psi} = \frac{1}{d^2}$$
$$\int_{\overline{\mathcal{M}}_{0,2}}\frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}$$

for simplicity of notation. Such an assumption will give the correct final results.

For a nonnegative integer g and a positive integer h, let $\overline{\mathcal{M}}^{\bullet}_{\chi,h}$ be the moduli of possibly disconnected stable curves C with h marked points such that

- If $C_1, \ldots, C_k$ are connected components of C, and g_i is the arithmetic genus of C_i, then
$$\sum_{i=1}^{k}(2-2g_i) = \chi.$$
- Each connected component contains at least one marked point.

The connected components of $\overline{\mathcal{M}}^{\bullet}_{\chi,h}$ are of the form
$$\overline{\mathcal{M}}_{g_1,h_1} \times \cdots \times \overline{\mathcal{M}}_{g_k,h_k}.$$
where
$$\sum_{i=1}^{k}(2-2g_i) = \chi, \quad \sum_{i=1}^{k} h_i = h.$$
The restriction of the Hodge bundle $\mathbb{E} \to \overline{\mathcal{M}}^{\bullet}_{\chi,h}$ to the above connected component is the direct sum of the Hodge bundles on each factor, and
$$\Lambda^{\vee}(u) = \prod_{i=1}^{k} \Lambda^{\vee}_{g_i}(u).$$

We define
$$\overline{\mathcal{M}}_{2l(\mu^+),\mu^+,\chi,\mu^-,2l(\mu^-)} = \overline{\mathcal{M}}^{\bullet}_{\chi,l(\mu^+)+l(\mu^-)}.$$
For $-\chi^{\pm} + l(\mu^{\pm}) + l(\nu^{\pm}) > 0$, we define
$$\begin{aligned}\overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\mu^-,2l(\mu^-)} &= \left(\overline{\mathcal{M}}^{\bullet}_{\chi^+}(\mathbb{P}^1,\mu^+,\nu^+)//\mathbb{C}^*\right) \times \overline{\mathcal{M}}_{\chi^0,l(\nu^+)+l(\mu^-)},\\ \overline{\mathcal{M}}_{2l(\mu^+),\mu^+,\chi^0,\nu^-,\chi^-} &= \overline{\mathcal{M}}_{\chi^0,l(\mu^+)+l(\nu^-)} \times \left(\overline{\mathcal{M}}^{\bullet}_{\chi^-}(\mathbb{P}^1,\nu^-,\mu^-)//\mathbb{C}^*\right),\end{aligned}$$
$$\begin{aligned}&\overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}\\ = &\left(\overline{\mathcal{M}}^{\bullet}_{\chi^+}(\mathbb{P}^1,\mu^+,\nu^+)//\mathbb{C}^*\right) \times \overline{\mathcal{M}}^{\bullet}_{\chi^0,l(\nu^+)+l(\nu^-)} \times \left(\overline{\mathcal{M}}^{\bullet}_{\chi^-}(\mathbb{P}^1,\nu^-,\mu^-)//\mathbb{C}^*\right).\end{aligned}$$

For every $(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-) \in G^{\bullet}_{\chi}(X,\mu^+,\mu^-)$, there is a morphism
$$i_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-} : \overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-} \to \overline{\mathcal{M}}^{\bullet}_{\chi}(X,\mu^+,\mu^-),$$
whose image $\mathcal{F}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}$ is a union of connected components of $\hat{\mathcal{M}}$. The morphism $i_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}$ induces an isomorphism
$$\overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}/A_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-} \cong \mathcal{F}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-},$$
where $A_{2l(\mu^+),\mu^+,\chi,\mu^-,l(\mu^-)}$ is trivial, and for $\chi^{\pm} + l(\mu^{\pm}) + l(\nu^{\pm}) > 0$, we have
$$1 \to \prod_{i=1}^{l(\nu^+)} \mathbb{Z}_{\nu_i^+} \to A_{\chi^+,\nu^+,\chi^0,\mu^-,2l(\mu^-)} \to \mathrm{Aut}(\nu^+) \to 1,$$
$$1 \to \prod_{j=1}^{l(\nu^-)} \mathbb{Z}_{\nu_j^-} \to A_{2l(\mu^+),\mu^+,\chi^0,\nu^-,\chi^-} \to \mathrm{Aut}(\nu^-) \to 1,$$
$$1 \to \prod_{i=1}^{l(\nu^+)} \mathbb{Z}_{\nu_i^+} \times \prod_{j=1}^{l(\nu^-)} \mathbb{Z}_{\nu_j^-} \to A_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-} \to \mathrm{Aut}(\nu^+) \times \mathrm{Aut}(\nu^-) \to 1.$$

So for $-\chi^{\pm}+l(\mu^{\pm})+l(\nu^{\pm})>0$, we have

$$|A_{\chi^+,\nu^+,\chi^0,\mu^-,2l(\mu^-)}|=z_{\nu^+},\quad |A_{2l(\mu^+),\mu^+,\chi^0,\nu^-,\chi^-}|=z_{\nu^-},$$

$$|A_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}|=z_{\nu^+}z_{\nu^-}.$$

The stack $\hat{\mathcal{M}}$ is a disjoint union of

$$\{\mathcal{F}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}:(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-)\in G^\bullet_\chi(X,\mu^+,\mu^-)\}.$$

7.3. **Contribution from each admissible label.** Let $N^{\mathrm{vir}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}$ on $\overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}$ be the pull back of the virtual normal bundle of $\mathcal{F}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}$ in $\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)$. Calculations similar to those in [21, Appendix A] show that

$$\frac{i^*_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}e_T(V^\bullet_{\chi,\mu^+,\mu^-})}{e_T(N^{\mathrm{vir}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-})}=A^+A^0A^-,$$

where

$$\begin{aligned}
A^0 &= (-1)^{|\nu^+|+|\nu^-|+1}a_{\nu^+}a_{\nu^-}\prod_{i=1}^{l(\nu^+)}\frac{\prod_{a=1}^{\nu_i^+-1}(\nu_i^+\beta+a\alpha)}{(\nu_i^+-1)!\alpha^{\nu_i^+-1}}\prod_{j=1}^{l(\nu^-)}\frac{\prod_{a=1}^{\nu_j^--1}(\nu_j^-\alpha+a\beta)}{(\nu_j^--1)!\beta^{\nu_j^--1}}\\
&\quad\cdot\frac{\Lambda^\vee(\alpha)\Lambda^\vee(\beta)\Lambda^\vee(-\alpha-\beta)(\alpha\beta(\alpha+\beta))^{l(\nu^+)+l(\nu^-)-1}}{\prod_{i=1}^{l(\nu^+)}\alpha(\alpha-\nu_i^+\psi_i)\prod_{j=1}^{l(\nu^-)}\beta(\beta-\nu_j^-\psi_{l(\nu^+)+j})}\\
A^+ &= \begin{cases}(-1)^{l(\mu^+)}, & \chi^-=2l(\mu^-)\\ (-1)^{-\frac{\chi^+}{2}+l(\nu^+)+l(\mu^+)}a_{\nu^+}\frac{\beta^{-\chi^++l(\mu^+)+l(\nu^+)}}{-\alpha-\psi^+}, & \text{otherwise}\end{cases}\\
A^- &= \begin{cases}(-1)^{l(\mu^-)}, & \chi^+=2l(\mu^+)\\ (-1)^{-\frac{\chi^-}{2}+l(\nu^-)+l(\mu^-)}a_{\nu^-}\frac{\alpha^{-\chi^-+l(\mu^-)+l(\nu^-)}}{-\beta-\psi^-}, & \text{otherwise}\end{cases}
\end{aligned}$$

From the definitions in Section 2.2 and Proposition 5.4, we have

$$\begin{aligned}
&I_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}(\alpha,\beta)\\
=\;&\frac{1}{|A_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}|}\int_{[\overline{\mathcal{M}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}]^{\mathrm{vir}}}\frac{i^*_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}e_T(V^\bullet_{\chi,\mu^+,\mu^-})}{e_T(N^{\mathrm{vir}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-})}\\
=\;&|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|\frac{\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}{(-1)^{|\mu^+|+|\mu^-|}}G^\bullet_{\chi^0,\nu^+,\nu^-}(\alpha,\beta)\\
&\cdot z_{\nu^+}\frac{(-\sqrt{-1}\beta/\alpha)^{-\chi^++l(\nu^+)+l(\mu^+)}}{(-\chi^++l(\nu^+)+l(\mu^+))!}H^\bullet_{\chi^+,\nu^+,\mu^+}\cdot z_{\nu^-}\frac{(-\sqrt{-1}\alpha/\beta)^{-\chi^-+l(\nu^-)+l(\mu^-)}}{(-\chi^-+l(\nu^-)+l(\mu^-))!}H^\bullet_{\chi^-,\nu^-,\mu^-}
\end{aligned}$$

Let $\tau=\beta/\alpha$. Then

$$\begin{aligned}
&I_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}(\alpha,\beta)\\
=\;&|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|\frac{\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}{(-1)^{|\mu^+|+|\mu^-|}}G^\bullet_{\chi^0,\nu^+,\nu^-}(\tau)\\
&\cdot z_{\nu^+}\frac{(-\sqrt{-1}\tau)^{-\chi^++l(\nu^+)+l(\mu^+)}}{(-\chi^++l(\nu^+)+l(\mu^+))!}H^\bullet_{\chi^+,\nu^+,\mu^+}\cdot z_{\nu^-}\frac{(-\sqrt{-1}/\tau)^{-\chi^-+l(\nu^-)+l(\mu^-)}}{(-\chi^-+l(\nu^-)+l(\mu^-))!}H^\bullet_{\chi^-,\nu^-,\mu^-}
\end{aligned}$$

7.4. **Sum over admissible labels.**

$$
\begin{aligned}
& K^\bullet_{\chi,\mu^+,\mu^-} \\
= & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(X,\mu^+,\mu^-)]^{\mathrm{vir}}} e_T(V^\bullet_{\chi,\mu^+,\mu^-}) \\
= & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\sum_{(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-)\in G^\bullet_\chi(X,\mu^+,\mu^-)} \frac{1}{|A_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}|} \\
& \cdot\int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}} \frac{i^*_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-} e_T(V^\bullet_{\chi,\mu^+,\mu^-})}{e_T(N^{\mathrm{vir}}_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-})} \\
= & \frac{1}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\sum_{(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-)\in G^\bullet_\chi(X,\mu^+,\mu^-)} I_{\chi^+,\nu^+,\chi^0,\nu^-,\chi^-}(\alpha,\beta) \\
= & \frac{\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}{(-1)^{|\mu^+|+|\mu^-|}}\Bigg(\sum_{(\chi^+,\nu^+,\chi^0,\nu^-,\chi^-)\in G^\bullet_\chi(X,\mu^+,\mu^-)} G^\bullet_{\chi^0,\nu^+,\nu^-}(\tau) \\
& \cdot z_{\nu^+}\frac{(-\sqrt{-1}\tau)^{-\chi^++l(\nu^+)+l(\mu^+)}}{(-\chi^++l(\nu^+)+l(\mu^+))!}H^\bullet_{\chi^+,\nu^+,\mu^+}\cdot z_{\nu^-}\frac{(-\sqrt{-1}/\tau)^{-\chi^-+l(\nu^-)+l(\mu^-)}}{(-\chi^-+l(\nu^-)+l(\mu^-))!}H^\bullet_{\chi^-,\nu^-,\mu^-}\Bigg)
\end{aligned}
$$

Recall that

$$
K^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{\chi\in 2\mathbb{Z},\chi\le 2(l(\mu^+)+l(\mu^-))} \lambda^{-\chi+l(\mu^+)+l(\mu^-)}\frac{(-1)^{|\mu^+|+|\mu^-|}}{\sqrt{-1}^{l(\mu^+)+l(\mu^-)}}K^\bullet_{\chi,\mu^+,\mu^-}.
$$

We have

$$
K^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu^\pm|=|\mu^\pm|}\Phi^\bullet_{\mu^+,\nu^+}(-\sqrt{-1}\tau\lambda)z_{\nu^+}G^\bullet_{\nu^+,\nu^-}(\lambda;\tau)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}\left(\frac{-\sqrt{-1}}{\tau}\lambda\right).
$$

This finishes the proof of (6).

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