MARIÑO-VAFA FORMULA AND HODGE INTEGRAL IDENTITIES

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1. Introduction

Based on string duality Mariño and Vafa [10] conjectured a closed formula on certain Hodge integrals in terms of representations of symmetric groups. This formula was first explicitly written down by the third author in [13] and proved in joint work [8] of the authors of the present paper. For a different approach see [12]. Our proof follows the strategy of proving both sides of the equation satisfy the same cut-and-join equation and have the same initial values.

In this note we will describe a proof of the ELSV formula relating Hurwitz numbers and Hodge integrals:

$$H_{g,\mu} = \frac{(2g-2+|\mu|+l(\mu))!}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

along the same lines. By the Burnside formula, the Hurwitz numbers are related to the representations of symmetric groups, hence so is the Hodge integral on the right-hand side of the above formula.

Mariño and Vafa have remarked that in principle it is possible to obtain all Hodge integrals involving up to three Hodge classes from their formula. Another purpose of this note is to show that our method is not only useful in proving the MV and ELSV formulas, but also is powerful in deriving some consequences from them. For example, as easy consequences of the MV formula and the cut-and-join equation, we will present unified simple proofs of the λ_g conjecture [3]

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\dots,k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1+\cdots+k_n = 2g-3+n$, and the following identities for Hodge integrals [2]:

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g\sum_{i=1}^{2g-1}\frac{1}{i} - \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1}b_{g_2}.$$

We also describe a method to compute general λ_{g-1} integrals. To summarize, our results in this work partly verify Mariño and Vafa's anticipation for the applications of their formula.

2. Preliminaries

2.1. **Mumford's relations.** Let $\mathbb{E}$ be the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$, and let $\lambda_i = c_i(\mathbb{E})$. Define

$$c_t(\mathbb{E}) = \sum_{i=0}^{g} t^i \lambda_i, \qquad \Lambda_g^\vee(t) = \sum_{i=0}^{g} (-1)^i \lambda_i t^{g-i}.$$

Then we have

$$c_{-t}(\mathbb{E}) = c_t(\mathbb{E}^\vee) = (-t)^g \Lambda_g^\vee(\frac{1}{t}).$$

Mumford's relations are given by:

$$c_t(\mathbb{E})c_{-t}(\mathbb{E}) = 1. \tag{1}$$

Equivalently,

$$\Lambda_g^\vee(t)\Lambda_g^\vee(-t) = (-1)^g t^{2g}. \tag{2}$$

2.2. **Some consequences.** From the following well-known relation between Newton polynomials and elementary symmetric polynomials (cf. e.g. [9]):

$$\sum_{k\geq 1} p_k t^{k-1} = \frac{E_n'(-t)}{E_n(-t)},$$

we get:

$$\sum_{n\geq 1} n! t^{n-1} \operatorname{ch}_n(\mathbb{E}) = \frac{c_{-t}'(\mathbb{E})}{c_{-t}(\mathbb{E})} = c_t(\mathbb{E}) c_{-t}'(\mathbb{E}). \tag{3}$$

It can be rewritten as

$$\sum_{n\geq 1} n! t^{n-1} \operatorname{ch}_n(\mathbb{E}) = \sum_{i=1}^{g} i\lambda_i (-t)^{i-1} \sum_{j=0}^{g} \lambda_j t^j.$$

Hence $\operatorname{ch}_k(\mathbb{E}) = 0$ for $k \geq 2g$, and

$$n! \operatorname{ch}_n(\mathbb{E}) = \sum_{i+j=n} (-1)^{i-1} i \lambda_i \lambda_j. \tag{4}$$

It is not hard to see that

$$\operatorname{ch}_{2m}(\mathbb{E}) = 0, \tag{5}$$

$$(2g-1)! \operatorname{ch}_{2g-1}(\mathbb{E}) = (-1)^{g-1} \lambda_{g-1}\lambda_g, \tag{6}$$

$$(2g-3)! \operatorname{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1}(3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2}). \tag{7}$$

We will need the following results:

Lemma 2.1.

$$\Lambda_g^\vee(1)(\Lambda_g^\vee)'(-1) = (-1)^{g-1} g + \sum_{k\geq 1} k!(-1)^{k-1} \operatorname{ch}_k(\mathbb{E}), \tag{8}$$

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \left(\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\right) = -\lambda_{g-1} + g\lambda_g - \lambda_g \sum_{k\geq 1} k!(-1)^{k-1} \operatorname{ch}_k(\mathbb{E}). \tag{9}$$

In particular the degree $3g-3$ part of the left-hand side of (9) is

$$(-1)^{g-1}\lambda_g\lambda_{g-1}\lambda_{g-2}.$$

Proof.

$$\Lambda_g^\vee(1)(\Lambda_g^\vee)'(-1) = \Lambda_g^\vee(1)\sum_{j=0}^{g}(-1)^j(g-j)t^{g-j-1}\lambda_j|_{t=-1}$$

$$\begin{aligned} &= (-1)^{g-1}g\Lambda_g^\vee(1)\sum_{j=0}^{g}\lambda_j + \Lambda_g^\vee(1)\cdot(-1)^g\sum_{j=0}^{g}j\lambda_j \\ &= (-1)^{g-1}g + (-1)^g c_{-t}(\mathbb{E})c_t'(\mathbb{E})|_{t=1} \\ &= (-1)^{g-1}g + (-1)^g\sum_{k\geq 1}k!(-1)^{k-1}\operatorname{ch}_k(\mathbb{E}). \end{aligned}$$

$$\begin{aligned} &\left.\frac{d}{d\tau}\right|_{\tau=0}\big(\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\big) \\ = \;& \Lambda_g^\vee(1)\left.\frac{d}{d\tau}\right|_{\tau=0}\Lambda_g^\vee(\tau)\cdot\Lambda_g^\vee(-1) + \Lambda_g^\vee(1)\Lambda_g^\vee(0)\cdot\left.\frac{d}{d\tau}\right|_{\tau=0}\Lambda_g^\vee(-\tau-1) \\ = \;& -\lambda_{g-1} - (-1)^g\lambda_g\Lambda_g^\vee(1)(\Lambda_g^\vee)'(-1) \\ = \;& -\lambda_{g-1} + g\lambda_g - \lambda_g\sum_{k\geq 1}k!(-1)^{k-1}\operatorname{ch}_k(\mathbb{E}). \end{aligned}$$

□

3. Cut-and-Join Equation, Mariño-Vafa Formula and ELSV Formula

We recall in this section the Mariño-Vafa formula recently proved in [8]. We also describe a proof of the ELSV formula along the same lines. Both of these formulas can be proved by the cut-and-join equation method. These formula relate the geometry of moduli spaces of Riemann surfaces encoded in Hodge integrals to combinatorics of the representations of symmetric groups, and hence to the theories of affine Kac-Moody Lie algebras and symmetric functions.

3.1. **Mariño-Vafa formula.** For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 0)$, define

$$\begin{aligned} \mathcal{C}_{g,\mu}(\tau) &= -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \\ &\quad\cdot\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}, \\ \mathcal{C}_\mu(\lambda;\tau) &= \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}\mathcal{C}_{g,\mu}(\tau). \end{aligned}$$

Note that

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}}\frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}}\frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu)\geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} > 0)$. Define the following generating series

$$\mathcal{C}(\lambda;\tau;p) = \sum_{|\mu|\geq 1} \mathcal{C}_\mu(\lambda;\tau)p_\mu.$$

This finishes the definitions of the geometric side of the Mariño-Vafa formula.

For the representation theoretical side, for a partition μ, denote by χ_μ the character of the irreducible representation of $S_{|\mu|}$ indexed by μ, and by $C(\mu)$ the conjugacy class of $S_{|\mu|}$ indexed by μ. Define:

$$\begin{aligned} V_\mu(\lambda) = &\prod_{1\leq a<b\leq l(\mu)} \frac{\sin\left[(\mu_a - \mu_b + b - a)\lambda/2\right]}{\sin\left[(b-a)\lambda/2\right]} \\ &\cdot \frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i} 2\sin\left[(v - i + l(\mu))\lambda/2\right]}. \end{aligned} \tag{10}$$

This has an interpretation in terms of quantum dimension [10]. Define

$$\begin{aligned} R(\lambda;\tau;p) = &\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \\ &\sum_\mu \left(\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} V_{\nu^i}(\lambda) \right) p_\mu. \end{aligned}$$

The Mariño-Vafa formula is:

$$\mathcal{C}(\lambda;\tau;p) = R(\lambda;\tau;p). \tag{11}$$

These formulas were explicitly written down in this form in [13]. In [7, 8] it is proved by showing $\mathcal{C}$ and R both satisfy the following cut-and-join equation:

$$\frac{\partial \Gamma}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left(ijp_{i+j}\frac{\partial^2\Gamma}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial\Gamma}{\partial p_i}\frac{\partial\Gamma}{\partial p_j} + (i+j)p_ip_j\frac{\partial\Gamma}{\partial p_{i+j}} \right), \tag{12}$$

and have the same initial values:

$$\mathcal{C}(\lambda;0;p) = R(\lambda;0;p) = -\sum_{d\geq 1} \frac{\sqrt{-1}^{d+1}p_d}{2d\sin(d\lambda/2)}. \tag{13}$$

3.2. **A proof of ELSV formula by cut-and-join equation.** Given a partition μ of length $l(\mu)$, denote by $H_{g,\mu}$ the Hurwitz numbers of almost simple Hurwitz covers of $\mathbb{P}^1$ of ramification type μ by connected genus g Riemann surfaces. The ELSV formula [1, 5] states:

$$H_{g,\mu} = \frac{(2g-2+|\mu|+l(\mu))!}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}. \tag{14}$$

We now describe how to prove this formula by the cut-and-join equation.

Define

$$\Phi_\mu(\lambda) = \sum_{g\geq 0} H_{g,\mu} \frac{\lambda^{2g-2+|\mu|+l(\mu)}}{(2g-2+|\mu|+l(\mu))!},$$
$$\Phi(\lambda,p) = \sum_\mu \Phi_\mu(\lambda)p_\mu,$$
$$\Psi_\mu(\lambda) = \sum_{g\geq 0} \frac{\lambda^{2g-2+|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$
$$\Psi(\lambda;p) = \sum_\mu \Psi_\mu(\lambda)p_\mu.$$

The $\Phi(\lambda;p)$ and $\Psi(\lambda;p)$ both satisfy the following cut-and-join equation:

$$\frac{\partial\Theta}{\partial\lambda} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2\Theta}{\partial p_i\partial p_j} + ijp_{i+j}\frac{\partial\Theta}{\partial p_i}\frac{\partial\Theta}{\partial p_j} + (i+j)p_ip_j\frac{\partial\Theta}{\partial p_{i+j}}\right). \tag{15}$$

For Φ this was proved in [4] by combinatorial method, and in [6] by symplectic method. For Ψ one can apply the method of [8]. The initial values can be easily determined as follows. It is easy to see that

$$H_{0,\mu} = \delta_{\mu,(1)},$$

hence

$$\Phi(0;p) = p_1.$$

On the other hand, since $2g-2+|\mu|+l(\mu) > 0$ unless $g=0$ and $\mu=(1)$, it is straightforward to see that

$$\Psi(0;p) = p_1.$$

Hence after transferring the cut-and-join equation to a sequence of systems of ODEs for e^Φ and e^Ψ, one sees that

$$e^\Phi = e^\Psi.$$

This proves the ELSV formula.

By the Burnside formula, one easily gets the following expression (see e.g. [13]):

$$\Phi(\lambda;p) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_\mu \sum_{\cup_{i=1}^n \mu_i=\mu}\prod_{i=1}^n \sum_{|\nu_i|=|\mu_i|}\frac{\chi_{\nu_i}(\mu_i)}{z_{\mu_i}} e^{\kappa_{\nu_i}\lambda/2}\frac{\dim R_{\nu_i}}{|\nu_i|!}p_\mu.$$

This reveals the close relationship between the Mariño-Vafa formula and the ELSV formula.

4. A Simple Proof Of The λ_g Conjecture

The following formula is called the λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\dots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}, \tag{16}$$

for $k_1+\cdots+k_n = 2g-3+n$, $g>0$. It is proved in [3] in a very complicated way. We will give in this section a simple proof based on the Mariño-Vafa formula and the cut-and-join equation.

4.1. **A reformulation of the λ_g conjecture.** We begin with the following reformulation:

Lemma 4.1. *the λ_g conjecture is equivalent to*

$$\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^n(1-\mu_i\psi_i)}=d^{n-3}\frac{d\lambda/2}{\sin(d\lambda/2)}, \tag{17}$$

for all partitions of d.

Proof. The left-hand side of (17) is

$$\begin{aligned}&\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^n(1-\mu_i\psi_i)}\\ =\ &\sum_{g\geq 0}\lambda^{2g}\sum_{k_1+\cdots+k_n=2g-3+n}\prod_{i=1}^n\mu_i^{k_i}\cdot\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{i=1}^n\psi_i^{k_i}.\end{aligned}$$

By (32) in the Appendix the right-hand side is:

$$\begin{aligned}&d^{n-3}\left(1+\sum_{g\geq 1}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}(d\lambda)^{2g}\right)\\ =\ &(\sum_i\mu_i)^{n-3}+\sum_{g\geq 1}\lambda^{2g}\sum_{\sum_i k_i=2g-3+n}\prod_{i=1}^n\mu_i^{k_i}\cdot\binom{2g+n-3}{k_1,\ldots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}.\end{aligned}$$

The Lemma is proved by comparing the coefficients. □

4.2. **A simple proof of the λ_g conjecture by the Mariño-Vafa formula.** The following result has been proved in [13] by the cut-and-join equation.

Theorem 4.2. *Write $R(\lambda;\tau;p)=\sum_\mu R_\mu(\lambda;\tau)p_\mu$. Then one has*

$$\begin{aligned}&\lim_{\tau\to 0}\lambda^{2-l(\mu)}\frac{1}{(\tau(\tau+1))^{l(\mu)-1}}\prod_{i=1}^{l(\mu)}\frac{(\mu_i-1)!}{\prod_{j=1}^{\mu_i-1}(j+\mu_i\tau)}\frac{\prod_j m_j(\mu)!}{\sqrt{-1}^{|\mu|+l(\mu)}}R_\mu(\lambda;\tau)\\ =&d^{l(\mu)-3}\cdot\frac{d\lambda/2}{\sin(d\lambda/2)}.\end{aligned} \tag{18}$$

By the Mariño-Vafa formula, the left-hand side of (18) is

$$\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(0)\Lambda_g^\vee(-1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}=\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\lambda_g}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}.$$

Therefore, we have established (17) hence proved the λ_g conjecture (16).

5. Derivation of Some Other Hodge Integral Identities

In this section we show how to derive from the Mariño-Vafa formula and the cut-and-join equation the following formula proved in [2] by different method:

$$\int_{\overline{\mathcal{M}}_g}\lambda_{g-2}\lambda_{g-1}\lambda_g=\frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}, \tag{19}$$

$$\int_{\overline{\mathcal{M}}_{g,1}}\frac{\lambda_{g-1}}{1-\psi_1}=b_g\sum_{i=1}^{2g-1}\frac{1}{i}-\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,g_2>0}}\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}b_{g_1}b_{g_2}, \tag{20}$$

where

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0. \end{cases}$$

5.1. **The derivative.** We begin with the following special case of the Mariño-Vafa formula:

$$\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{1-d\psi_i} = -\frac{\lambda}{\sqrt{-1}^{d+1}} \frac{(d-1)!}{\prod_{a=1}^{d-1}(d\tau+a)} R_{(d)}(\lambda;\tau). \tag{21}$$

Lemma 5.1.

$$\left.\frac{d}{d\tau}\right|_{\tau=0} R_{(d)}(\lambda;\tau) = \sum_{i+j=d, i\neq j} \frac{-\sqrt{-1}^{d+1}\lambda}{8\sin(i\lambda/2)\sin(j\lambda/2)}. \tag{22}$$

Proof. This is an easy consequence of the cut-and-join equation:

$$\begin{aligned} \left.\frac{d}{d\tau}\right|_{\tau=0} R_{(d)}(\lambda;\tau) &= \frac{\sqrt{-1}\lambda}{2} \sum_{i+j=d} \big(ijR_{(i,j)}(\lambda;0) + ijR_{(i)}(\lambda;0)R_{(j)}(\lambda;0)\big) \\ &= \frac{\sqrt{-1}\lambda}{2} \sum_{i+j=d} ij \frac{-\sqrt{-1}^{i+1}}{2i\sin(i\lambda/2)} \frac{-\sqrt{-1}^{j+1}}{2j\sin(j\lambda/2)} \\ &= \sum_{i+j=d} \frac{-\sqrt{-1}^{d+1}\lambda}{8\sin(i\lambda/2)\sin(j\lambda/2)}. \end{aligned}$$

□

Corollary 5.1. *We have*

$$\begin{aligned} &\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\frac{d}{d\tau}\big|_{\tau=0} \big(\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\big)}{1-d\psi_1} \\ &= -\sum_{a=1}^{d-1} \frac{1}{a} \cdot \frac{d\lambda/2}{d\sin(d\lambda/2)} + \sum_{i+j=d} \frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)}. \end{aligned} \tag{23}$$

Proof. Take derivative in τ and set $\tau = 0$ on both sides of equation (21). By (22) we get

$$\begin{aligned} &\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\frac{d}{d\tau}\big|_{\tau=0} \big(\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\big)}{1-d\psi_1} \\ = & -\frac{\lambda}{\sqrt{-1}^{d+1}} \left.\frac{d}{d\tau}\right|_{\tau=0} \frac{(d-1)!}{\prod_{a=1}^{d-1}(d\tau+a)} \cdot R_{(d)}(\lambda;0) - \frac{\lambda}{\sqrt{-1}^{d+1}} \left.\frac{d}{d\tau}\right|_{\tau=0} R_{(d)}(\lambda;\tau) \\ = & -\sum_{a=1}^{d-1} \frac{1}{a} \cdot \frac{d\lambda/2}{d\sin(d\lambda/2)} + \sum_{i+j=d} \frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)}. \end{aligned}$$

□

Now the left-hand side of (22) is a polynomial in d hence so must be the right-hand side. If we find explicit expressions for the right-hand side, then by comparing the coefficients, we get Hodge integral identities. This is how we prove (19) and (20).

5.2. **The right-hand side.** We have

$$-\sum_{a=1}^{d-1}\frac{1}{a}\cdot\frac{d\lambda/2}{d\sin(d\lambda/2)}=-\sum_{a=1}^{d-1}\frac{1}{a}\sum_{g\geq 0}b_g d^{2g-1}\lambda^{2g}$$

hence the coefficient of λ^{2g} is

$$-\sum_{a=1}^{d-1}\frac{1}{a}\cdot b_g d^{2g-1}. \tag{24}$$

This cancels with a similar term from the second term on the right-hand side of (23).

We also have

$$\begin{aligned}&\sum_{i+j=d}\frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)}\\ =&\sum_{i+j=d}\frac{1}{2ij}\sum_{g_1\geq 0}b_{g_1}(i\lambda)^{2g_1}\cdot\sum_{g_2\geq 0}b_{g_2}(j\lambda)^{2g_2}\\ =&\frac{1}{2}\sum_{g\geq 0}\lambda^{2g}\sum_{g_1+g_2=g}b_{g_1}b_{g_2}\sum_{i+j=d}i^{2g_1-1}j^{2g_2-1}.\end{aligned}$$

By (34) in the Appendix we have for $g_1, g_2 > 0$, and $g_1+g_2=g$,

$$\begin{aligned}&F_{g_1,g_2}(d)=\sum_{i+j=d}i^{2g_1-1}j^{2g_2-1}=\sum_{i=1}^{d-1}i^{2g_1-1}(d-i)^{2g_2-1}\\ =&\sum_{k=0}^{2g_2-1}(-1)^{2g_2-1-k}\binom{2g_2-1}{k}d^k\sum_{i=1}^{d-1}i^{2g_1+2g_2-2-k}\\ =&\sum_{k=0}^{2g_2-1}(-1)^{2g_2-1-k}\binom{2g_2-1}{k}d^k\sum_{l=0}^{2g-2-k}\frac{\binom{2g-1-k}{l}}{2g-1-k}B_l d^{2g-1-k-l}\\ =&\sum_{k=0}^{2g_2-1}\sum_{l=0}^{2g-2-k}\frac{(-1)^{2g_2-1-k}}{2g-1-k}\binom{2g_2-1}{k}\binom{2g-1-k}{l}B_l d^{2g-1-l}.\end{aligned}$$

It is easy to see that the coefficient of d in $F_{g_1,g_2}(d)$ receives contribution only from the term with $k=0$ and $l=2g-2$, hence it is

$$-B_{2g-2}.$$

The coefficient of d^{2g-1} in $F_{g_1,g_2}(d)$ receives contributions from terms with $l=0$, hence it is given by:

$$\sum_{k=0}^{2g_2-1}\frac{(-1)^{2g_2-1-k}}{2g-1-k}\binom{2g_2-1}{k}.$$

We can deal with the case of $g_1 = 0$ or $g_2 = 0$ in the same fashion.

$$F_{0,g}(d) = \sum_{i+j=d} i^{-1} j^{2g-1} = \sum_{i=1}^{d-1} i^{-1}(d-i)^{2g-1}$$

$$= \sum_{k=0}^{2g-1} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{i=1}^{d-1} i^{2g-2-k}$$

$$= \sum_{k=0}^{2g-2} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{l=0}^{2g-k} \frac{\binom{2g-1-k}{l}}{2g-1-k} B_l d^{2g-1-k-l} + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}$$

$$= \sum_{k=0}^{2g-2} \sum_{l=0}^{2g-2-k} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{l} B_l d^{2g-1-l} + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}.$$

The coefficient of d in $F_{0,g}(d)$ or $F_{g.0}(d)$ is

$$-B_{2g-2}.$$

The coefficient of d^{2g-1} in $F_{0,g}(d)$ or $F_{g,0}(d)$ is given by:

$$\sum_{k=0}^{2g-2} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} = \sum_{k=1}^{2g-1} \frac{(-1)^i}{i} \binom{2g-1}{i}.$$

5.3. **Proof of (19).** By Lemma 2.1, the coefficient of $d\lambda^{2g}$ of the left-hand side of (23) is:

$$(-1)^{g-1} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1 \lambda_g \lambda_{g-1} \lambda_{g-2} = (-1)^{g-1}(2g-2) \int_{\overline{\mathcal{M}}_g} \lambda_g \lambda_{g-1} \lambda_{g-2}. \tag{25}$$

By the above discussions, the coefficient of $d\lambda^{2g}$ on the right-hand side of (23) is

$$\frac{-B_{2g-2}}{2} \sum_{g_1+g_2=g} b_{g_1} b_{g_2} = \frac{-B_{2g-2}}{2} \cdot \frac{|B_{2g}|}{2g} \frac{1}{(2g-2)!}$$

Comparing with (25) we get

$$\int_{\overline{\mathcal{M}}_g} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{(-1)^g B_{2g-2}}{2(2g-2)} \cdot \frac{|B_{2g}|}{2g} \frac{1}{(2g-2)!}$$

This is exactly (19).

5.4. **Proof of (20).** The coefficient of $d^{2g-1}\lambda^{2g}$ on the left-hand side of (23) is

$$-\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-1} \lambda_{g-1} = -\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1}.$$

By the above discussions, it is equal to

$$b_g \sum_{i=1}^{2g-1} \frac{(-1)^i}{i} \binom{2g-1}{i} + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,g_2>0}} b_{g_1} b_{g_2} \sum_{k=0}^{2g_2-1} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k}.$$

Hence (20) is proved by the following:

Lemma 5.2.

$$\sum_{k=0}^{2g_2-1} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} = \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!},$$
$$\sum_{i=1}^{2g-1} \frac{(-1)^i}{i} \binom{2g-1}{i} = -\sum_{i=1}^{2g-1} \frac{1}{i}.$$

Proof. Let

$$f(x) = \sum_{i=1}^{2g-1} \frac{(-1)^i}{i} \binom{2g-1}{i} x^i.$$

Then we have

$$f'(x) = \sum_{i=1}^{2g-1} (-1)^i \binom{2g-1}{i} x^{i-1} = \frac{(1-x)^{2g-1}-1}{x} = -\sum_{i=0}^{2g-2} (1-x)^i$$

Hence we have

$$\sum_{i=1}^{2g-1} \frac{(-1)^i}{i} \binom{2g-1}{i} = \int_0^1 f'(x)dx = -\int_0^1 \sum_{i=0}^{2g-2} (1-x)^i dx = -\sum_{i=1}^{2g-1} \frac{1}{i}.$$

Similarly, let

$$g(x) = \sum_{k=0}^{2g_2-1} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} x^{2g-1-k}.$$

Then we have

$$g'(x) = \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} \binom{2g_2-1}{k} x^{2g-2-k} = x^{2g_1-1}(1-x)^{2g_2-1}.$$

Hence we have by integrations by parts:

$$\begin{aligned}
&\sum_{k=0}^{2g_2-1} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \\
= \;& \int_0^1 g'(x)dx = \int_0^1 x^{2g_1-1}(1-x)^{2g_2-1}dx \\
= \;& \frac{(2g_2-1)}{2g_1} \int_0^1 x^{2g_1}(1-x)^{2g_2-2}dx \\
= \;& \frac{(2g_2-1)(2g_2-2)}{2g_1(2g_1+1)} \int_0^1 x^{2g_1+1}(1-x)^{2g_2-3}dx \\
= \;& \cdots = \frac{(2g_2-1)!}{2g_1(2g_1+1)\cdots(2g_1+2g_2-1)} = \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}.
\end{aligned}$$

□

5.5. **Computation of the λ_{g-1} integrals.** We now describe how to generalize the above method to compute integrals of form

$$\int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\lambda_{g-1}}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

for a partition μ of d with $l(\mu) > 1$. We rewrite the Mariño-Vafa formula as follows.

$$
\begin{aligned}
&\sum_{g\geq 0} \lambda^{2g}(\tau+1)^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
&= -\frac{|\operatorname{Aut}(\mu)|\lambda^{l(\mu)-2}}{\sqrt{-1}^{|\mu|+l(\mu)}} \frac{R_\mu(\lambda;\tau)}{\tau^{l(\mu)-1}}.
\end{aligned} \tag{26}
$$

Now we take derivative in τ and then set $\tau = 0$. The left-hand side is given by

$$
\begin{aligned}
&\sum_{g\geq 0} \lambda^{2g}(l(\mu)-1) \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-1)\Lambda_g^\vee(0)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
&+\sum_{g\geq 0} \lambda^{2g} \sum_{i=1}^{l(\mu)} \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-1)\Lambda_g^\vee(0)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
&+\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\frac{d}{d\tau}\big|_{\tau=0}\left(\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)\right)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
= \quad & (l(\mu)-1)d^{l(\mu)-3}\frac{d\lambda/2}{\sin(d\lambda/2)} + \sum_{i=1}^{l(\mu)}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a} d^{l(\mu)-3}\frac{d\lambda/2}{\sin(d\lambda/2)} \\
&+\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{-\lambda_{g-1}+g\lambda_g-\lambda_g\sum_{k\geq 1}k!(-1)^{k-1}\operatorname{ch}_k(\mathbb{E})}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
= \quad & (l(\mu)-1)d^{l(\mu)-3}\frac{d\lambda/2}{\sin(d\lambda/2)} + \sum_{i=1}^{l(\mu)}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a} d^{l(\mu)-3}\frac{d\lambda/2}{\sin(d\lambda/2)} \\
&-\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\lambda_{g-1}}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} + \sum_{g\geq 0} g\lambda^{2g} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\lambda_g}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\
&-\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\lambda_g\sum_{k\geq 1}k!(-1)^{k-1}\operatorname{ch}_k(\mathbb{E})}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}.
\end{aligned}
$$

The last term can be computed by using Mumford's GRR relations [11]. The right-hand side can be computed by L'Hospital's rule:

$$
\begin{aligned}
&\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\left(\frac{R_\mu(\lambda;\tau)}{\tau^{l(\mu)-1}}\right) = \lim_{\tau\to 0}\frac{\tau\partial_\tau R_\mu(\lambda;\tau)-(l(\mu)-1)R_\mu(\lambda;\tau)}{\tau^{l(\mu)}} \\
= \quad & \frac{\partial_\tau^{l(\mu)}[\tau\partial_\tau R_\mu(\lambda;\tau)-(l(\mu)-1)R_\mu(\lambda;\tau)]|_{\tau=0}}{l(\mu)!} \\
= \quad & \frac{[\tau\partial_\tau^{l(\mu)+1}R_\mu(\lambda;\tau)+\partial_\tau^{l(\mu)}R_\mu(\lambda;\tau)]|_{\tau=0}}{l(\mu)!} = \frac{\partial_\tau^{l(\mu)}R_\mu(\lambda;\tau)|_{\tau=0}}{l(\mu)!},
\end{aligned}
$$

hence it can be found by applying the cut-and-join equation repeatedly and by the initial value of R. Therefore, one gets a method to compute

$$
\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\lambda_{g-1}}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}.
$$

Appendix A. Bernoulli numbers

In this Appendix we recall some well known facts about Bernoulli numbers. These numbers are defined by the following series expansion:

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{27}$$

The first few terms are given by

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_3 = 0.$$

For odd $m > 1$, $B_m = 0$, and for even m, the sign of B_{2n} is $(-1)^{n-1}$.

Lemma A.1. *For $m > 0$,*

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = 0. \tag{28}$$

Proof. Multiply both sides of (27) by e^t. The left-hand side becomes

$$e^t \frac{t}{e^t-1} = t + \frac{t}{e^t-1} = t + \sum_{m=0}^{\infty} B_m \frac{t^m}{m!};$$

the right-hand side becomes

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \cdot \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} B_k \frac{t^m}{k!(m-k)!}.$$

Hence for $m > 1$, we have

$$\frac{B_m}{m!} = \sum_{k=0}^{m} \frac{B_k}{k!(m-k)!}.$$

(28) follows easily. □

Lemma A.2.

$$\frac{t/2}{\sinh(t/2)} = \sum_{m=0}^{\infty} \frac{1-2^{m-1}}{2^{m-1}} \frac{B_m}{m!} t^m, \tag{29}$$

$$\frac{t}{2}\coth(t/2) = \sum_{n=0}^{\infty} B_{2n} \frac{t^{2n}}{(2n)!}. \tag{30}$$

Proof. These can be proved by easy algebraic manipulations as follows.

$$\begin{aligned}
\frac{t/2}{\sinh(t/2)} &= \frac{t}{e^t-1} e^{t/2} = 2\frac{t/2}{e^{t/2}-1} - \frac{t}{e^t-1} \\
&= 2\sum_{m=0}^{\infty} B_m \frac{(t/2)^m}{m!} - \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{1-2^{m-1}}{2^{m-1}} \frac{B_m}{m!} t^m.
\end{aligned}$$

$$\frac{t}{2}\coth(t/2) = \frac{t}{2}\frac{e^t+1}{e^t-1} = \frac{1}{2} + \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_{2n} \frac{t^{2n}}{(2n)!}.$$

□

Corollary A.1.

$$\sum_{i+j=n} \frac{1-2^{1-2i}}{(2i)!} B_{2i} \cdot \frac{1-2^{1-2j}}{(2j)!} B_{2j} = \frac{(1-2n)B_{2n}}{(2n)!}. \tag{31}$$

Proof. Apply the operator $t\frac{d}{dt}$ on both sides of (30):

$$\frac{t}{2}\coth\frac{t}{2} - \left(\frac{t/2}{\sinh(t/2)}\right)^2 = \sum_{n=1}^{\infty} B_{2n}\frac{t^{2n}}{(2n-1)!}.$$

Hence

$$\left(\frac{t/2}{\sinh(t/2)}\right)^2 = \frac{t}{2}\coth\frac{t}{2} - \sum_{n=0}^{\infty} B_{2n}\frac{t^{2n}}{(2n-1)!} = \sum_{n=0}^{\infty}(1-2n)B_{2n}\frac{t^{2n}}{(2n)!}.$$

From this (31) easily follows. □

By changing t to $\sqrt{-1}t$, one gets from (29) by recalling $B_{2n} = (-1)^{n-1}|B_{2n|}$:

$$\frac{t/2}{\sin(t/2)} = 1 + \sum_{g\geq 1} \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}t^{2g}. \tag{32}$$

And (31) becomes

$$\sum_{g_1+g_2=g} \frac{2^{2g_1-1}-1}{2^{2g_1-1}}\frac{|B_{2g_1}|}{(2g_1)!}\frac{2^{2g_2-1}-1}{2^{2g_2-1}}\frac{|B_{2g_2}|}{(2g_2)!} = \frac{|B_{2g}|}{2g}\frac{1}{(2g-2)!} \tag{33}$$

Finally recall for any positive integer m,

$$\sum_{i=1}^{d-1} i^m = \sum_{k=0}^{m} \frac{\binom{m+1}{k}}{m+1} B_k d^{m+1-k}. \tag{34}$$

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