Strictly Nonblocking Multirate $\log_d(N, m, p)$ Networks*

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Abstract

We give necessary and sufficient conditions for the d-nary multilog network to be strictly nonblocking under the discrete multirate model, and sufficient conditions for the same under the continuous multirate model.

Keywords: Strictly nonblocking network, multirate model, multilog network, Cantor network

1 Introduction

In a multirate network, each link has a (normalized) capacity 1 and each request for connection is associated with a weight (bandwidth requirement) w. Many paths can go through a link simultaneously as long as their total weight has not exceeded unity. In particular, an input or output (link) can generate or receive many requests as long as their total weight does not exceed unity. Often, the weight of a request is bounded in the range $[b, B]$. A more general model is to assume that an input or output link has capacity $\beta \leq 1$ to reflect the reality that many networks need an internal-to-external speed-up to be more efficient. In the discrete case (the channel model), we assume that each internal link has f_1 channels, each input or output has $f_0 \leq f_1$ channels, and a request is associated with a positive integer number $q, 1 \leq q \leq Q$, where $Q \leq f_0$ is an upper bound of the number of channels a request can demand.

A network state is a set of paths connecting a set of requests $\{(i_x, o_y, w)\}$ such that no link carries a load exceeding 1 (or f_1), where i_x is an input, o_y is an output and w is the associated weight. Given a state, a new request (i, o, w) must satisfy the condition that i has not generated and o has not received requests whose total weights are more than $1 - w$

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(or $f_0 - w$). A network is strictly nonblocking if at any state a new request can always be connected without any link carrying a load exceeding 1 (or f_1).

Strictly nonblocking multirate networks have been studied for the 3-stage Clos network and the d-nary Cantor network [1], $d \geq 2$. In this paper, we extend the results on the Cantor network to the more general $\log_d(N,m,p)$ network (also called the d-nary multilog network), where the Cantor network is the special case with $m = n-1$. In particular, we give necessary and sufficient conditions for $\log_d(N,m,p)$ to be multirate strictly nonblocking.

2 The channel model

The $\log_d(N,m,p)$ network was first proposed by Shyy and Lea [6], extending the $\log_d(N,0,p)$ network proposed by Lea [4]. The $\log_d(N,m,p)$ network has an input (output) stage consisting of $N = d^n$ $1 \times p$ ($p \times 1$) crossbars, and p copies of d-nary m-extra-stage, $1 \leq m \leq n-1$, inverse banyan network $BY_d^{-1}(n,m)$, where each input and output crossbar is connected to every copy of $BY_d^{-1}(n,m)$. Figure 1 illustrates an example of $\log_d(N,m,p)$.

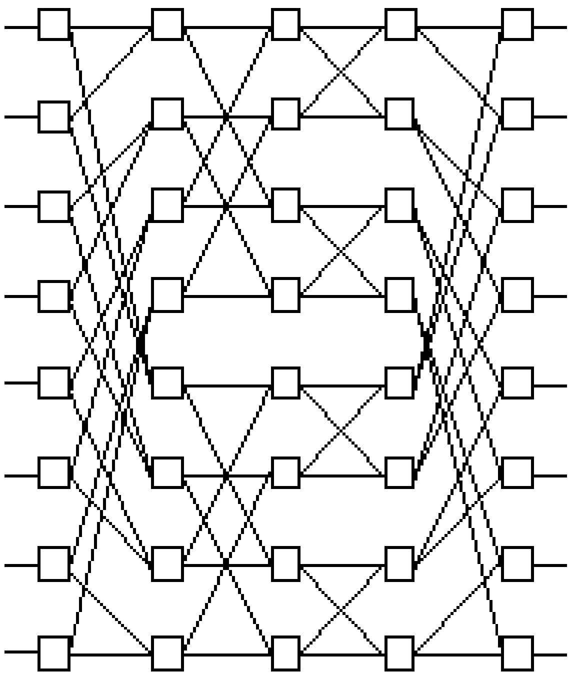

Figure 1: A $\log_2(8,0,2)$ network

We study the channel model in this section.

Theorem 2.1. *Consider the (Q, f_0, f_1) channel model with $d^{\lfloor \frac{n-1}{2} \rfloor} f_0 \geq f_1 + 1$. Then*

$\log_d(N,0,p)$ *is multirate strictly nonblocking if and only if*

$$p \geq \lfloor \frac{d^{\lfloor \frac{n-1}{2} \rfloor} f_0 - Q}{f_1 - Q + 1} \rfloor + \lfloor \frac{d^{\lceil \frac{n-1}{2} \rceil} f_0 - Q}{f_1 - Q + 1} \rfloor + 1.$$

Proof: Suppose the new request is (i,o,q). Call an internal link *q-saturated* if it always carries a load exceeding $f_1 - q + 1$ (hence cannot carry a new q-request). Note that the channel graph between i and o is just a single path l. An intersecting path is a path p from input $i' \neq i$ to output $o' \neq o$ such that p shares a link with l. In particular, let l_i denote the stage-i link (a link between stage i and stage $i+1$) in l. Then an i-intersecting path is one which intersects l at l_i. Note that an i-intersecting path can also be a j-intersecting path for $i \neq j$.

Sufficiency. The new request cannot be carried in a copy of $BY_d^{-1}(n,0)$ if and only if there exists an l_i which is q-saturated by i-intersecting paths. We divide l_i into two disjoint halves:

$$H_1 = \{l_i : 1 \leq i \leq \lfloor (n-1)/2 \rfloor\};$$

$$H_2 = \{l_i : \lfloor (n+1)/2 \rfloor \leq i \leq n-1\}.$$

Note that only $d^{\lfloor (n-1)/2 \rfloor}$ inputs can generate i-intersecting paths for i in H_1. The total weight of these paths is bounded by $d^{\lfloor (n-1)/2 \rfloor} f_0 - q$ since the weight of the new request must be excluded. In the right-hand side of the inequality in Theorem 2.1, the first term is an upper bound of the number of q-saturated l_i for i in H_1. Similarly, only $d^{\lceil (n-1)/2 \rceil}$ outputs can generate i-intersecting paths for i in H_2, and the second term in the inequality is an upper bound of the number of q-saturated l_i for i in H_2. Thus their sum is an upper bound of the number of saturated links in l, hence an upper bound of the number of blocked copies of $BY_d^{-1}(n,0)$. One more copy suffices to route the new request.

Necessity. With respect to l, an input (output) is called *i-marginal* if it can generate an i-intersecting path but not an $(i-1)$-intersecting ($(i+1)$-intersecting) path. Then there are d^i i-marginal inputs and d^{n-i} i-marginal outputs. Note that

$$d^i \leq d^{n-i} \qquad \text{for } i \text{ in } H_1,$$

$$d^i \geq d^{n-i} \qquad \text{for } i \text{ in } H_2.$$

Compute the maximum number b_1 of q-saturated links generated by 1-marginal inputs. Assign a total weight of $b_1(f_1-q+1)$ of requests to 1-marginal outputs (doable by the above inequalities), and mix the remaining requests of weight $df_0 - q - b_1(f_1-q+1)$ with requests generated by 2-marginal inputs. Again, compute the maximum number b_2 of q-saturated links generated by this mixture of requests. Assign requests with a total weight $b_2(f_1-q+1)$ to 2-marginal outputs, and mix the rest with requests from 3-marginal inputs. Proceed like this until the last step $s = \lfloor (n-1)/2 \rfloor$. At step s, assign requests with a total weight of $b_s(f_1-q+1)$ to s-marginal outputs and ignore unassigned requests. It is straightforward to verify that the number of saturated l_i for i in H_1 constructed by this assignment is the first term in the inequality of Theorem 2.1. Similarly, the corresponding number for i in H_2 is the second term. Thus their sum plus one copy is the necessary number of copies to route the new request. $\square$

Next we consider the general m case. Define

$$g_m(q) = \sum_{j=1}^{m} \frac{1}{d^j} \{ \lfloor \frac{d^j f_0 - q}{f_1 - q + 1} \rfloor - \lfloor \frac{d^{j-1} f_0 - q}{f_1 - q + 1} \rfloor \} \qquad \text{for} \quad 1 \leq m \leq n-1.$$

Theorem 2.2 *Consider the* (Q, f_0, f_1) *channel model with* $d^{\lfloor \frac{n-1}{2} \rfloor} f_0 \geq f_1 + 1$. *Then* $\log_d(N, m, p)$ *is multirate strictly nonblocking for* $0 \leq m \leq n-1$ *if and only if*

$$p \geq \lfloor g_m(Q) + \lfloor \frac{d^{\lfloor \frac{n+m-1}{2} \rfloor} f_0 - Q - \lfloor \frac{d^m f_0 - Q}{f_1 - q + 1} \rfloor (f_1 - Q + 1)}{f_1 - Q + 1} \rfloor \frac{1}{d^m} \rfloor$$

$$+ \lfloor g_m(Q) + \lfloor \frac{d^{\lceil \frac{n+m-1}{2} \rceil} f_0 - Q - \lfloor \frac{d^m f_0 - Q}{f_1 - Q + 1} \rfloor (f_1 - Q + 1)}{f_1 - Q + 1} \rfloor \frac{1}{d^m} \rfloor \} + 1.$$

Proof: The $m = 0$ case was proved in Theorem 2.1 and the $m = n-1$ (the Cantor network) case in [1]. We prove the general m case.

The strategy is to partition the network stages into two parts: the outer part consists of m outer stages from both the input side and the output side; and the inner part consists of $n - m$ inner stages composed by d^m copies of $BY_d^{-1}(n-m, 0)$. For the outer part, we adopt (and extend) the approach of Chung and Ross [2] given for the special case of Cantor network. For the inner part we apply Theorem 2.1.

More specifically, for the outer part, we compute the total weight of requests which can reach a stage$-j$ link, $1 \leq j \leq (n+m-1)/2$, in the (i, o) channel graph to be $d^j f_0 - q$, while a q-saturated link carries a load at least $f_1 - q + 1$. Thus at most

$$\lfloor \frac{d^j f_0 - q}{f_1 - q + 1} \rfloor$$

links in the channel graph at or before stage j can be saturated. The worst case is to assign the saturated links to as early a stage as possible since links in the early stages have more blocking power. This results in assigning

$$\lfloor \frac{d^j f_0 - q}{f_1 - q + 1} \rfloor$$

saturated links to stage j, each of which blocks $1/d^j$ copies of a $BY_d^{-1}(n, m)$. Thus $\lfloor g_m(q) \rfloor$ is the number of copies of $BY_d^{-1}(n, m)$ blocked by paths intersecting the links of the (i, o) channel graph in the first or last m stages.

The total weight of request which can reach at or before a stage$-j$ link, $1 \leq j \leq \lfloor (n+m-1)/2 \rfloor$, is

$$d^{\lfloor \frac{n+m-1}{2} \rfloor} f_0 - q.$$

But a total weight of

$$\lfloor \frac{d^m f_0 - q}{f_1 - q + 1} \rfloor (f_1 - q + 1)$$

was already connected in the first m stages. Therefore only the difference of these two weights can be used to saturate stage$-j$ links for $m < j \leq n$. Since the $\log_d(N, m, p)$ network between these stages consists of d^m copies of $BY_d^{-1}(n-m, 0)$, we apply Theorem 2.1 (only the input side) to the number of copies of $BY_d^{-1}(n-m, 0)$ blocked, which must be divided by d^m to convert to the number of copies of $BY_d^{-1}(n, m)$ blocked.

The argument for the output side is analogous. One extra channel then guarantees the routing of the current request. Therefore

$$p \geq \max_{1 \leq q \leq Q} \{ \lfloor g_m(q) + \lfloor \frac{d^{\lfloor \frac{n+m-1}{2} \rfloor} f_0 - q - \lfloor \frac{d^m f_0 - q}{f_1 - q + 1} \rfloor (f_1 - q + 1)}{f_1 - q + 1} \rfloor \frac{1}{d^m} \rfloor$$

$$+ \lfloor g_m(q) + \lfloor \frac{d^{\lceil \frac{n+m-1}{2} \rceil} f_0 - q - \lfloor \frac{d^m f_0 - q}{f_1 - q + 1} \rfloor (f_1 - q + 1)}{f_1 - q + 1} \rfloor \frac{1}{d^m} \rfloor \} + 1 \tag{1}$$

is a sufficient condition for $\log_d(N, m, p)$ being multirate strictly nonblocking for $0 \leq m \leq n-1$. We show the maximum is achieved at $q = Q$. To see it, define

$$A_j = \lfloor \frac{d^j f_0 - q}{f_1 - q + 1} \rfloor, \qquad j = 1, \cdots, \lfloor \frac{n+m-1}{2} \rfloor$$

for simplicity. Then trivially $A_0 = \lfloor \frac{f_0 - q}{f_1 - q + 1} \rfloor = 0$ and $A_j \geq 0$ is nondecreasing in q for every $j = 0, 1, \cdots, \lfloor \frac{n+m-1}{2} \rfloor$. We have

$$\begin{aligned}
& g_m(q) + \lfloor \frac{d^{\lfloor \frac{n+m-1}{2} \rfloor} f_0 - q - \lfloor \frac{d^m f_0 - q}{f_1 - q + 1} \rfloor (f_1 - q + 1)}{f_1 - q + 1} \rfloor \frac{1}{d^m} \\
= \; & g_m(q) + (\lfloor \frac{d^{\lfloor \frac{n+m-1}{2} \rfloor} f_0 - q}{f_1 - q + 1} \rfloor - \lfloor \frac{d^m f_0 - q}{f_1 - q + 1} \rfloor) \frac{1}{d^m} \\
= \; & \frac{1}{d}(A_1 - A_0) + \frac{1}{d^2}(A_2 - A_1) + \cdots + \frac{1}{d^m}(A_m - A_{m-1}) + \frac{1}{d^m}(A_{\lfloor \frac{n+m-1}{2} \rfloor} - A_m) \\
= \; & (\frac{1}{d} - \frac{1}{d^2}) A_1 + (\frac{1}{d^2} - \frac{1}{d^3}) A_2 + \cdots + (\frac{1}{d^{m-1}} - \frac{1}{d^m}) A_{m-1} + \frac{1}{d^m} A_{\lfloor \frac{n+m-1}{2} \rfloor}.
\end{aligned}$$

Since every term is positive and nondecreasing in q, we conclude that the first term in the right-hand side of (1) is maximized at Q. A similar conclusion holds for the second term. It follows that (1) is maximized at Q.

The necessity part follows from the fact that the conditions of Chung and Ross and of Theorem 2.1 are both necessary. □

We further study the situation where the internal links have different capacities. Suppose the input and output have capacity f_0, the stage-i links and stage-$(n-i)$ links have capacity f_i, $i = 1, 2, \ldots, \lceil \frac{n+m-1}{2} \rceil$, and $f_{i-1} \leq f_i$.

We define

$$l_k(q) = \lfloor \frac{d^k f_0 - q - \sum_{i=1}^{k-1}(f_i - q + 1) l_i(q)}{f_k - q + 1} \rfloor \qquad \text{for} \quad 1 \leq k \leq \lceil \frac{n+m-1}{2} \rceil,$$

and

$$g_m(q) = \sum_{j=1}^{m} \frac{1}{d^k} l_k(q) \qquad \text{for} \quad 1 \leq m \leq n-1.$$

Theorem 2.3 *Consider the* $(Q, f_0, f_1, \cdots, f_{\lceil \frac{n+m-1}{2} \rceil})$ *channel model with* $d^{\lfloor \frac{n-1}{2} \rfloor} f_0 \geq f_1 + 1$. *Then* $\log_d(N, m, p)$ *is multirate strictly nonblocking for* $0 \leq m \leq n-1$ *if and only if*

$$p \geq \max_{1 \leq q \leq Q} \{ \lfloor g_m(q) + \frac{1}{d^m} \sum_{m+1}^{\lfloor \frac{n+m-1}{2} \rfloor} l_k(q) \rfloor + \lfloor g_m(q) + \frac{1}{d^m} \sum_{m+1}^{\lceil \frac{n+m-1}{2} \rceil} l_k(q) \rfloor \} + 1.$$

Proof: The proof is analogous to the proof of Theorem 2.2. The assumption $f_{i-1} \leq f_i$ for $i = 1, 2, \cdots, \lceil \frac{n+m-1}{2} \rceil$ is needed to guarantee

$$\frac{f_0}{f_k - q + 1} \cdot \frac{1}{d^k} > \frac{f_0}{f_{k+1} - q + 1} \cdot \frac{1}{d^{k+1}} \qquad \text{for} \quad 1 \leq k \leq m,$$

$$\frac{f_0}{f_k - q + 1} \cdot \frac{1}{d^m} > \frac{f_0}{f_{k+1} - q + 1} \cdot \frac{1}{d^m} \qquad \text{for} \quad 1 \leq k \leq \lfloor \frac{n+m-1}{2} \rfloor,$$

such that if an intersecting path intersects at several stages, the blocking effect is always greatest at the outmost stage, justifying our assigning it to that stage. □

3 The continuous model

We first quote a lemma proved by Melen and Turner [5].

Lemma 3.1 $\lfloor \frac{a-w}{b-w+\epsilon} \rfloor = \lceil \frac{a-w}{b-w} \rceil - 1$ *if* $a \geq b$, *where* ϵ *is positive and tends to* 0.

Theorem 3.2 *Consider the* (b, B, β) *continuous model satisfying* $d^{\lfloor \frac{n-1}{2} \rfloor} \beta > 1$. *Then* $\log_d(N, 0, p)$ *is strictly nonblocking if*

$$p \geq \lceil \frac{d^{\lfloor \frac{n-1}{2} \rfloor} \beta - B}{1 - B} \rceil + \lceil \frac{d^{\lceil \frac{n-1}{2} \rceil} \beta - B}{1 - B} \rceil - 1,$$

and the condition is necessary if $b \leq \frac{1}{k} < B$ *for some integer* k.

Proof: With an argument analogous to the proof of Theorem 2.1, we obtain the sufficient condition to route an (i, o, w) new request to be

$$\begin{aligned} & \lfloor \frac{d^{\lfloor \frac{n-1}{2} \rfloor} \beta - w}{1 - w + \epsilon} \rfloor + \lfloor \frac{d^{\lceil \frac{n-1}{2} \rceil} \beta - w}{1 - w + \epsilon} \rfloor + 1 \\ = \quad & \lceil \frac{d^{\lfloor \frac{n-1}{2} \rfloor} \beta - w}{1 - w} \rceil + \lceil \frac{d^{\lceil \frac{n-1}{2} \rceil} \beta - w}{1 - w} \rceil - 1, \end{aligned}$$

which is maximized at $w = B$.

To prove necessity, note that the combination of $k-1$ weights of $\frac{1}{k}$ and one weight of $\frac{1}{k}+\epsilon$ constitute a total weight of $1+\epsilon$. Therefore if $w = \frac{1}{k}$, then we can have every saturated link carrying a load of $1 - \frac{1}{k} + \epsilon$. □

If the condition $b \le \frac{1}{k} < B$ is not met, say $\frac{1}{k+1} < b < B < \frac{1}{k}$, then every internal link can carry a maximum of k connections. If further, $\frac{\beta}{k+1} < b < B < \frac{\beta}{k}$, then this can be treated as the channel model and Theorem 2.1 applies with $Q = 1$.

Theorem 3.3 *Consider the (b, B, β) continuous model satisfying $b+B \ge 1$. Then $\log_d(N, 0, p)$ is strictly nonblocking if and only if*

$$p \ge \lfloor \frac{\beta}{b} \rfloor (d^{\lfloor \frac{n-1}{2} \rfloor} - 1) + \lfloor \frac{\beta}{b} \rfloor (d^{\lceil \frac{n-1}{2} \rceil} - 1) + 1.$$

Proof: Let the new request be (i, o, w). Note that the channel graph between i and o is just a single path l. Similar to the proof of Theorem 2.1, $d^{\lfloor (n-1)/2 \rfloor} - 1$ inputs other than i can generate i-intersecting paths for i in H_1, and $d^{\lceil (n-1)/2 \rceil} - 1$ outputs can generate i-intersecting paths for i in H_2. Since either an input or an output can generate at most $\lfloor \beta/b \rfloor$ requests,

$$\lfloor \frac{\beta}{b} \rfloor (d^{\lfloor \frac{n-1}{2} \rfloor} - 1) + \lfloor \frac{\beta}{b} \rfloor (d^{\lceil \frac{n-1}{2} \rceil} - 1)$$

is an upper bound of the number of intersecting paths, hence an upper bound of the number of blocked copies. Thus one extra copy suffices to carry the new request. Note that whether the extra copy carries any load from i or o is immaterial since the load cannot exceed $\beta - w$.

On the other hand, suppose $w = B$. Then the worst case described above can happen and the new request cannot be routed through any link already carrying a load b. Hence the sufficient condition is also necessary. □

Next we consider $m > 0$ case. Define

$$g_m(w) = \sum_{j=1}^{m} \frac{1}{d^j} \{ \lceil \frac{d^j \beta - w}{1-w} \rceil - \lceil \frac{d^{j-1}\beta - w}{1-w} \rceil \} \qquad \text{for} \quad 1 \le m \le n-1.$$

Theorem 3.4 *Consider the (b, B, β) continuous model satisfying $d^{\lfloor \frac{n-1}{2} \rfloor}\beta > 1$. Then $\log_d(N, m, p)$ is strictly nonblocking if*

$$p \ge \lfloor g_m(B) + \lceil \frac{d^{\lfloor \frac{n+m-1}{2} \rfloor}\beta - B - (\lceil \frac{d^m\beta - B}{1-B} \rceil - 1)(1-B)}{(1-B)} \rceil \frac{1}{d^m} \rfloor$$

$$+ \lfloor g_m(B) + \lceil \frac{d^{\lceil \frac{n+m-1}{2} \rceil}\beta - B - (\lceil \frac{d^m\beta - B}{1-B} \rceil - 1)(1-B)}{(1-B)} \rceil \frac{1}{d^m} \rfloor - 1,$$

and the condition is necessary if $b \le \frac{1}{k} < B$ for some integer k.

Proof: Analogous to the proof of Theorem 2.2. □

Theorem 3.5 *Consider the (b, B, β) continuous model satisfying $b+B > 1$. Then $\log_d(N, m, p)$ is strictly nonblocking if and only if*

$$p \geq 2\lfloor\frac{m(d-1)\lfloor\beta/b\rfloor}{d}\rfloor + \lfloor\frac{(d^{\lfloor\frac{n+m-1}{2}\rfloor}-1)\lfloor\beta/b\rfloor}{d^m}\rfloor + \lfloor\frac{(d^{\lceil\frac{n+m-1}{2}\rceil}-1)\lfloor\beta/b\rfloor}{d^m}\rfloor + 1.$$

Proof: Each input can generate at most $\lfloor\beta/b\rfloor$ requests. Let each internal link carry at most one request. Then there are $d^j - d^{j-1}$ inputs generating $(d^j - d^{j-1})\lfloor\beta/b\rfloor$ requests to intersect a stage$-j$ link in the (i, o)-channel graph for $1 \leq j \leq m$. Since each such intersecting path blocks $1/d^j$ copies of $BY_d^{-1}(n, m)$, they block a total of

$$\lfloor\sum_{j=1}^{m}\frac{(d^j - d^{j-1})\lfloor\beta/b\rfloor}{d^j}\rfloor = \lfloor\frac{m(d-1)\lfloor\beta/b\rfloor}{d}\rfloor$$

copies. Similarly, the output side blocks the same number of copies. Finally, stage $m+1$ to stage $n-m-1$ consists of d^m copies of $BY_d^{-1}(n-m, 0)$. We use an argument analogous to the proof of Theorem 2.2 to compute the number of copies blocked in these stages to be

$$\lfloor\frac{(d^{\lfloor\frac{n+m-1}{2}\rfloor}-1)\lfloor\beta/b\rfloor}{d^m}\rfloor + \lfloor\frac{(d^{\lceil\frac{n+m-1}{2}\rceil}-1)\lfloor\beta/b\rfloor}{d^m}\rfloor.$$

So one extra copy suffices to route the new request.

To prove necessity, let $w = B$. Then the worst case discussed above can happen. □

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