Semiclassical Limit of the Gross-Pitaevskii Equation in an Exterior Domain

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June 30, 2003

Abstract

In this paper, we study the semiclassical limit of the Gross-Pitaevskii equation (a cubic nonlinear Schrödinger equation) with the Neumann boundary condition in an exterior domain. We prove that before the formation of singularities in the limit system, the quantum density and the quantum momentum converge to the unique solution of the compressible Euler equation with the slip boundary condition as the scaling parameter approaches 0.

Mathematics Subject Classification (1995): 35L65, 81

Keywords: Semiclassical limit, Schrödinger equation, compressible Euler equation.

1 Introduction

Here we consider the local in time semi-classical limit of the Gross-Pitaevskii equation (a cubic Schrödinger equation) in the exterior of two dimensional domain in $\mathbb{R}^2$. More precisely, let Ω be an exterior domain in $\mathbb{R}^2$ such that $\partial\Omega$ is a bounded, smooth curve, and let $\nu(x)$ be the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. We study the following equations when the parameter ϵ goes to zero:

$$\begin{cases} i\,\epsilon\,\partial_t\,\psi^\epsilon = -\dfrac{\epsilon^2}{2}\,\Delta\psi^\epsilon + \left(|\psi^\epsilon|^2 - 1\right)\psi^\epsilon, & \text{in } \Omega\times\mathbb{R}_+ \\ \psi^\epsilon(t=0,x) = \sqrt{\rho_0^\epsilon(x)}\,\exp\left(\dfrac{i}{\epsilon}S_0^\epsilon(x)\right), & \\ \dfrac{\partial\psi^\epsilon}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad \psi^\epsilon(t,x)\to\exp\left(\dfrac{i}{\epsilon}\,S^\infty(x)\right) & \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

where ϵ is a small positive parameter, $S^\infty(x) = u^\infty \cdot x$, and u^∞ is a constant two-vector.

The motivation to study the problem (1.1) comes from many intresting issues concerning a superfluid passing an obstacle, see for example [FPR] and [JP]. The nonlinear Schrödinger equation (0.1), which is also called the Gross-Pitaevskii equation, has been proposed and studied as the fundament equation for understanding superfluids, see Ginzburg-Pitaevskii [GP], Landau-Lifschitz [LL], Gross [G] and many others. It has also been used to model phenomena in the Bose-Einstein condensates. The model mathematical problem for a superfluid passing an obstacle is as follows:

$$\begin{aligned} &-i\,u_t = \Delta u + u(1-|u|^2) \quad \text{ in } \mathbb{R}^2/\mathbb{B}_R \\ &\text{with } \left.\frac{\partial u}{\partial \nu}\right|_{\partial \mathbb{B}_R} = 0 \quad \text{and} \quad u(x,0) \approx e^{i\,v_0\cdot x} \end{aligned} \tag{1.2}$$

at $|x| = +\infty$. Here $\mathbb{B}_R$ denotes the obstacle. Since in (1.2), one has normalized the equation in such a way the Planck constant becomes 1. Thus the size R is often much larger than the unity. The well-known Madelung transform (see [M]) is to introduce two real variables $\rho \geq 0$ and ϕ such that $u = \sqrt{\rho}\, e^{i\,\phi}$. Then under a suitable condition one can show that (1.2) is equivalent to the fluid-type equations.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \text{div }(\rho u) = 0 \\ \dfrac{\partial}{\partial t}\,(\rho u) + \text{div}\,(\rho\,(u \otimes u)) + \nabla \left(\dfrac{\rho^2}{2}\right) = \rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \end{cases} \tag{1.3}$$

Here $u = \nabla\phi$. We note also that the phase dynammics according to

$$\frac{\partial \phi}{\partial t} = \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - |\nabla\phi|^2 + (1-\rho). \tag{1.4}$$

The term on the right-hand side of the second equation in (1.3) is called the quantum pressure. It can be formally argued that this quantum pressure term can neglected in a limiting process when the obstacle size $\mathbb{B}_R$ (or R) is much larger compared with the microscopic scale of the Gross-Pitaevskii equation (which is normalized to be 1), and when one is interested in only "long-wave" approximations (see [FPR]). Indeed, set $R = \frac{1}{\epsilon}$, and consider $\psi_\epsilon(x,t) = \sqrt{\rho^\epsilon(x,t)}\; e^{\frac{i}{\epsilon}\,S^\epsilon(x,t)}$ with $\nabla\, S^\epsilon(x,\bullet) \simeq u^\infty$ at $|x| = \infty$, then after a proper scaling of spatial and time-variables, one reduces to study (1.1) and its associated fluid type equation:

$$\begin{cases} \partial_t \rho^\epsilon + \text{div}(\rho^\epsilon u^\epsilon) = 0 \\ \partial_t(\rho^\epsilon\, u^\epsilon) + \text{div}(\rho^\epsilon\, u^\epsilon \otimes u^\epsilon) + \dfrac{1}{2}\,\nabla(\rho^\epsilon)^2 = \dfrac{\epsilon^2}{2}\,\rho^\epsilon \nabla\left(\dfrac{\Delta\sqrt{\rho^\epsilon}}{\sqrt{\rho^\epsilon}}\right) \\ \rho^\epsilon(t=0,x) = \rho_0^\epsilon(x), \quad u^\epsilon(t=0,x) = \nabla S_0^\epsilon(x) \end{cases} \tag{1.5}$$

where $u^\epsilon = \nabla S^\epsilon$. The domain Ω is now given by $\mathbb{R}^2 | B_1$, and the boundary conditions can be written in the following equivalent form:

$$\begin{cases} \epsilon \dfrac{\partial \sqrt{\rho^\epsilon}}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u^\epsilon \cdot \nu \Big|_{\partial\Omega} = 0, \qquad \text{and} \quad \rho^\epsilon(t,x) \to 1, \\ u(t,x) \to u^\infty \qquad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.6}$$

Thus, the formal WKB-limit as $\epsilon \to 0$ of (1.5)–(1.6) is given by the following compressible Euler equation:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \dfrac{1}{2} \nabla \rho^2 = 0 \\ u(t=0,x) = u_0(x), \quad \rho(t=0,x) = \rho_0(x) \end{cases} \tag{1.7}$$

with the slip boundary condition.

$$u \cdot \nu \Big|_{\partial\Omega} = 0, \ \text{ and } \ \rho(t,x) \to 1, \quad u(t,x) \to u^\infty \ \text{ as } \ |x| \to \infty. \tag{1.8}$$

Of course, it is necessary to assume that $(\rho_0^\epsilon(x), \nabla S_0^\epsilon(x))$ converges to $(\rho_0(x), u_0(x))$ in some appropriate sense. It should be noted that the first boundary condition in (1.6), that is, $\epsilon \frac{\partial}{\partial \nu}\left(\sqrt{\rho\,\epsilon}\right)|_{\partial\Omega} = 0$, disappers in the limiting process $\epsilon \to 0^+$. Otherwise it would lead to an additional boundary condition for the limit system (1.7) which would be undesirable.

When $\Omega = \mathbb{R}^d$ and if there is no super fluid at the infinity, the nonlinear term $(|\psi^\epsilon|^2 - 1)\psi^\epsilon$ in (1.1) is often replaced by $g(|\psi^\epsilon|^2)\psi^\epsilon$ with $g'(\cdot) > 0$. If, in addition, the phase function S^ϵ is independent of ϵ, and the amplitude is given by the expansion: $\sum_{j=1}^N a_j(x)\epsilon^j + \epsilon^N r_N(x,\epsilon)$ with $\lim_{\epsilon\to 0} \|r_N(\cdot,\epsilon)\|_{H^s} = 0$ for s large enough, Grenier ([Grenier98]) obtained a similar expansion for the solution of (1.1) in a small time. His main idea is that: instead of looking, as usual ,for solutions ψ^ϵ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x) e^{i\frac{S(t,x)}{\epsilon}} \tag{1.9}$$

with $S(t,x)$ independent of ϵ, $a^\epsilon(t,x)$ a real valued function, he looks for solutions ψ^ϵ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x)\, e^{i\frac{S^\epsilon(t,x)}{\epsilon}} = \left(a_1^\epsilon(t,x) + i a_2^\epsilon(t,x)\right) e^{i\frac{S^\epsilon(t,x)}{\epsilon}} \tag{1.10}$$

with $a_1^\epsilon, a_2^\epsilon, S^\epsilon$ being real valued functions. By plugging (1.10) into (1.1), separating the real and imaginary part, one can get the governing equations for $a_1^\epsilon, a_2^\epsilon$ and ∇S^ϵ. Then the standard energy estimate for symmetric hyperbolic system can be used to solve the resulting problem. But unfortunately, this method can not be applied here. The main difficulty lies in the Neumann boundary condition in (1.1). In fact, if we assume the solution has the form (1.10), then the boundary condition $\frac{\partial \psi^\epsilon}{\partial \nu}|_{\partial\Omega} = 0$ can be rewritten in the following equivalent form:

$$\left(\epsilon \frac{\partial a_1^\epsilon}{\partial \nu} - a_2^\epsilon \frac{\partial S^\epsilon}{\partial \nu}\right)\Bigg|_{\partial\Omega} = 0, \quad \left(\epsilon \frac{\partial a_2^\epsilon}{\partial \nu} + a_1^\epsilon \frac{\partial S^\epsilon}{\partial \nu}\right)\Bigg|_{\partial\Omega} = 0.$$

With these nonlinear boundary conditions, all known existing methods for the energy estimates do not seem to work.

On the other hand,suppose $\Omega = \mathbb{R}^d$, if there is no super fluid at the infinity, and if the nonlinear Schrödinger equation in (1.1) is replaced by Schrödinger-Poisson equation (or slightly more general nonlinearity as that in [ZP2]), the second author uses Wigner measure and modifies the modulated energy estimate, the latter was introduced by Brenier in [Brenier2000] in the studying of the convergence of the scaled Vlasov-Poisson system to the incompressible Euler system, to prove the convergence of the quantum density and quantum momentum to the solution of compressible Euler equations before the formation of singularities in the limit system as ϵ approaches 0. Indeed, let $f^\epsilon(t, x, \xi)$ be the Wigner transform of $\psi^\epsilon(t, x)$, the main ingredient in [ZP1] (and slightly different one in [ZP2]) is to study the evolution of the following functional:

$$H^\epsilon(t) = \int_{\mathbb{R}^d} h^\epsilon(t, x)\, dx = \int_{\mathbb{R}^d} \frac{1}{2} \left(\int_{\mathbb{R}^d} |\xi - u(t, x)|^2 f^\epsilon(t, x, \xi)\, d\xi + |\nabla \triangle^{-1} (\rho^\epsilon - \rho)|^2 \right) dx, \tag{1.11}$$

where (ρ, u) is the unique local smooth solution to the limit system. Here, since we work on the exterior domain Ω, we even do not know how to appropriately modify the definition of the Wigner transform in Ω. Hence (1.11) cannot be directly applied. Fortunately, we observe that by (3.24) in [ZP1] one has

$$\int_{\mathbb{R}^d} |\xi - u(t, x)|^2 f^\epsilon(t, x, \xi)\, d\xi = |(\epsilon \nabla_x - iu)\, \psi^\epsilon|^2.$$

In other words what really was used in [ZP1] (or [ZP2]) is in fact $\int_{\mathbb{R}^d} h^\epsilon(t, x)\, dx$ with

$$h^\epsilon(t) =: \frac{1}{2} \left(|(\epsilon \nabla_x - iu)\, \psi^\epsilon|^2 + \left|\nabla \Delta^{-1} (\rho^\epsilon - \rho)\right|^2 \right).$$

In this paper, we shall consider even more direct and simpler functional:

$$H^\epsilon(t) =: \frac{1}{2} \int_\Omega |(\epsilon \nabla_x - i\, u)\, \psi^\epsilon|^2\, dx + \frac{1}{2} \int_\Omega |\rho^\epsilon - \rho|^2\, dx$$

It can be viewed as a defect measure in studying weakly convergent sequences of solutions. We shall prove that $H^\epsilon(t)$ satisfies a Gronwall-type growth estimate. Thus, if $H^\epsilon(0) \to 0$ as $\epsilon \to 0^+$, then $H^\epsilon(t) \to 0$ for t in an interval of considerations.

This argument can actually be used also to simplify part of proofs in [ZP1] and [ZP2]. It also avoids the use of much more sophisticated analytic tool–Wigner measures.

It should be mentioned that a similar idea was also used in a recent work [MP1] to study the quasi-neutral limit of the scaled Schrödinger-Poisson equation to the incompressible Euler equation in a periodic domain.

Before the presentation of the main result of this paper, let us first make the following assumptions :

(A1) $\left(\sqrt{\rho_0^\epsilon(x)}\, \exp\left(\frac{i}{\epsilon} S_0^\epsilon(x)\right) - \exp(\frac{iu^\infty \cdot x}{\epsilon}) \in H^3(\Omega)\right)$, and $\nabla \sqrt{\rho_0^\epsilon(x)}$, $\exp\left(\frac{i}{\epsilon}\, S_0^\epsilon\right) - \exp\left(\frac{iu^\infty \cdot x}{\epsilon}\right)$, $\sqrt{\rho_0^\epsilon(x)}\, \nabla S_0^\epsilon(x)$ are uniformly bounded in $L^2(\Omega)$;

(A2) both $\rho_0^\epsilon(x) - \rho_0(x)$ and $\sqrt{\rho_0^\epsilon(x)}\,(\nabla S_0^\epsilon(x) - u_0(x))$ converge to 0 in $L^2(\Omega)$.

To guarantee the local existence of smooth solution to the (1.7) and (1.8), we need the following compatibility conditions for the initial data:

(A3) let $\frac{1}{2} \leq \rho_0(x)$, $(\rho_0(x) - 1, u_0(x) - u^\infty) \in H^3(\Omega)$, then $\nu \cdot \partial_t^k u(0)\big|_{\partial\Omega} = 0$, $0 \leq k \leq 2$, with $\partial_t^k u(0)$ the k^{th} time derivative at $t = 0$ of any solution of (1.7) and (1.8).These derivatives can be calculated from the second equation of (1.7) to yield a condition in terms of ρ_0 and u_0.

Here is our main Theorem:

Theorem 1.1. *Let the initial datum $(\rho_0^\epsilon(x), S^\epsilon(x)), (\rho_0(x), u_0(x))$ satisfy (A1–A3). $\psi^\epsilon(t,\ x), (\rho(t,x), u(t,x))$ be the solutions to* (1.1) *and* (1.7)*–*(1.8) *respectively. Then there exists a positive constant T^* such that for all $T < T^*$, $(\rho(t,x) - 1, u(t,x) - u^\infty) \in \bigcap_{j=0}^{2} C^j([0,T], H^{3-j}(\Omega))$, furthermore,*

$$|\psi^\epsilon(t,x)|^2 - \rho(t,x) \to 0 \ \text{in}\ L^\infty\left([0,T], L^2(\Omega)\right), \tag{1.12}$$

$$\epsilon \mathit{Im}\left(\overline{\psi^\epsilon}(t,x)\nabla\psi^\epsilon(t,x)\right) \to (\rho u)(t,x)\ \text{in}\ L^\infty\left([0,T], L^1_{loc}(\Omega)\right), \tag{1.13}$$

as $\epsilon \to 0$.

Remark 1.1.

1) Comparing the above Theorem with the results in [ZP1] and [ZP2], we improved the convergence in (1.13). In [ZP1] and [ZP2], one only proved: for any fixed $t < T^*$, there holds

$$\epsilon \mathrm{Im}(\overline{\psi^\epsilon(t,\cdot)}\nabla\psi^\epsilon(t,\cdot)) \rightharpoonup (\rho u)(t,\cdot) \text{ in the sense of measure.}$$

2) By modifying the proof a little bit, we can show Theorem 1.1 for a more general nonlinearity and in exterior domain of general space dimension. For a clear presentation, we are not going to pursue that here.

Finally, we would like to point out that for the classical fluids, it is well-known (see [DD]) there is a critical speed v_0 of the fluids at infinity such that whenever $|u^\infty| < v_0$, there is a steady state solution of (1.7). More precisely, there is a smooth solution of

$$\mathrm{div}(\rho\,\nabla\phi) = 0 \ \text{ in } \Omega, \quad \nabla\phi_{(\infty)} = u^\infty, \tag{1.14}$$

with $\rho = 1 - |\nabla\phi|^2 > 0$ in Ω (see also (1.4)). Solutions of (1.14) have maximum of $|\nabla\phi|$ achieved somewhere on $\partial\Omega$.

On the other hand, when $|u^\infty| > v_0$, then there is no smooth solution to (1.14). The flow (1.7) with such initial data would develop shock in a later time.

One often refer to the former case as subsonic and the later case as supersonic. One consequence, of our convergence theorem (1.1) for the semiclassical limit, is that in this limiting process the same picture remains valid in the subsonic case. Since a superfluid is by definition frictionless, there cannot be shock waves developed for (1.1). (In particular the flow (1.1) is time reversible.) What would be the substitution for "shock" has been addressed in [FPR] and [JP]. However, a precisely mathematical proof has not been found particularly for the case of transonic, that is when $|u^\infty| \simeq v_0$.

2 The local existence of smooth solution to (1.7)-(1.8)

In this section, we will prove the local existence of smooth solution to the exterior problem of the limit system (1.7-1.8). Actually we will study the problem with more general pressure term than that in (1.7):

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \qquad x \in \Omega, \quad t \geq 0, \\ \partial_t u + u \cdot \nabla u + \nabla P(\rho) = 0, \\ (\rho(t=0,x), u(t=0,x)) = (\rho_0(x), u_0(x)), \end{cases} \tag{2.1}$$

with the boundary conditions:

$$u \cdot \nu \,|_{\partial\Omega} = 0, \quad \rho(t,x) \to \rho^\infty, u \to u^\infty, \text{ as } |x| \to \infty. \tag{2.2}$$

To guarantee the strict hyperbolicity of (2.1), we need the assumption that

$$P'(\cdot) > 0. \tag{2.3}$$

When $\rho^\infty = 0$, $u^\infty = 0$ and Ω is a bounded domain, this problem has been studied by Beirao in [Bei81] and [Bei92]. And the local existence of smooth solutions to the full ideal gas dynamics equations in a bounded domain has been studied by Schochet in [Sch86]. In this section, we are going to modify the arguments in [Bei81], [Bei92] and [Sch86] to yield the local well-posedness of (2.1–2.2).

For convenience, let us denote $\bigcap_{j=0}^k C^j([0,T], H^{k-j}(\Omega))$ by $X_{k,T}$, with the norm $|||w|||_{k,T} = \sup_{0\leq t\leq T} |||w(t)|||_k$ and $|||w(t)|||_k = \sum_{j=0}^k \|\partial_t^j w(t,\cdot)\|_{H^{k-j}(\Omega)}$. As a convention in this section, $C(\cdot,\cdot,\cdots)$ will be constants, which are nondecreasing functions of their variables and they may change from line to line.

Then the following Theorem is the main result of this section:

Theorem 2.1. *Let $(\rho_0(x) - \rho^\infty, u_0(x) - \bar{u}(x)) \in H^3(\Omega)$, and satisfy the compatibility condition (A3) in the introduction, where $\bar{u}(x) \in C^\infty(\Omega)$, with*

$$\bar{u}(x) = \begin{cases} 0, & \text{if } x \in \{x : |x| \leq R\}, \\ u^\infty, & \text{if } x \in \{x : |x| \geq 2R\}, \end{cases}$$

for a sufficiently large R so that $\Omega \subset \{x : |x| \leq R\}$. Then there exists a positive constant T^*, such that (2.1–2.2) has a unique local smooth solution (ρ, u) with $(\rho(t,x) - \rho^\infty, u(t,x) - \bar{u}(x)) \in X_{3,T}$, for any $T < T^*$.

Remark 2.1. It should be noted here that with more smooth initial data and along with compatibility conditions, we can get a more smooth solution. And the proof of the Theorem is not only for space dimension 2, but it works for general space dimension greater than 1.

Proof. Motivated by [Bei81] and [Bei92], let us denote $g(t,x) =: \log \frac{\rho(t,x)}{\rho^\infty}$, then (2.1) can be written in the following equivalent form:

$$\begin{cases} \partial_t g + u \cdot \nabla g + \mathrm{div} u = 0, \qquad x \in \Omega, \quad t \geq 0, \\ \partial_t u + u \cdot \nabla u + \nabla h(g) = 0, \\ (g(t=0,x), u(t=0,x)) = (g_0, u_0), \end{cases} \tag{2.4}$$

together with the boundary conditions:

$$u \cdot \nu \mid_{\partial\Omega} = 0, \quad g(t,x) \to 0, \quad u(t,x) \to u^\infty, \text{ as } |x| \to \infty, \tag{2.5}$$

where $h(g) = P(\rho^\infty e^g)$. Then by (2.3), we have $h'(g) > 0$. The initial condition that $(\rho_0(x) - \rho^\infty, u_0(x) - \bar{u}(x)) \in H^3(\Omega)$ is changed to $(g_0(x), u_0(x) - \bar{u}(x)) \in H^3(\Omega)$.

To construct the approximate solutions, let us first smooth the initial data. As in [RM74] and [Sch86], we can obtain (g_0^n, u_0^n) such that $(g_0^n(x), u_0^n(x) - \bar{u}(x)) \in H^5(\Omega)$ obeying the compatibility conditions for (2.4) and (2.5) up to the order three, and converging in H^3 to $(g_0(x), u_0(x) - \bar{u}(x))$ as $n \to \infty$. In particular, from the second equation of (2.4), $\partial_t^k u^n(0,x) \in H^{5-k}(\Omega), 0 \leq k \leq 5$. Let $T_0 > 0$, by Theorem 2.5.7 of [H63], there exist functions $\tilde{u}^n(t,x)$ with $\tilde{u}^n(t,x) - \bar{u}(x) \in H^5([0,T_0] \times \Omega)$ satisfying $\partial_t^k \tilde{u}^n(0,x) = \partial_t^k u^n(0,x),\ 0 \leq k \leq 4$.

Now let us extend $\nu(x)$ to be in $C^\infty(\overline{\Omega})$ As in [Sch86], we define the approximate solutions (g^n, u^n) through the following equations:

$$\begin{cases} \partial_t g^n + u^n \cdot \nabla g^n + \mathrm{div} u^n = 0, \qquad x \in \Omega, \quad t \geq 0, \\ \partial_t u^n + u^n \cdot \nabla u^n + \epsilon(\nu \cdot \nabla) u^n + \nabla h(g^n) = \epsilon(\nu \cdot \nabla)\tilde{u}^n, \\ (g^n(t=0,x), u^n(t=0,x)) = (g_0^n, u_0^n), \end{cases} \tag{2.6}$$

together with the boundary condition (2.5).

Then the boundary is non-characteristic for this system, the boundary condition is maximally nonnegative, and compatibility conditions are satisfied up to the order three (see [RM74] and [Sch86] for more details). Hence by modifying the arguments in the Appendix of [Sch86] or [RM74], and the energy estimates, we can conclude that: there is a positive constant $T_{\epsilon,n}$ such that (2.6–2.5) has a unique solution (g^n, u^n) with $(g^n(t,x), u^n(t,x) - \bar{u}(x)) \in X_3\,([0, T_{\epsilon,n}] \times \Omega)$. We are going to prove that: there exists a n-independent positive constant T and a $\tilde{\epsilon}(n)$ such that if $\epsilon \leq \tilde{\epsilon}(n)$ and if n large enough, $T_{\epsilon,n} \geq T$ and

$$|||(g^n, u^n - \bar{u})|||_{3,T} \leq C, \tag{2.7}$$

for some constant C independent of ϵ and n.

We observe that (2.6) can also be written in the following form:

$$\partial_t w^n + \sum_{l=1}^{2} A^l(w^n)\,\partial_l w^n = L(w^n), \tag{2.8}$$

where

$$w^n = \begin{pmatrix} g^n \\ u_n^1 \\ u_n^2 \end{pmatrix}, \quad A(w^n,\xi) = \sum_{l=1}^{2} \xi_l\, A^l(w^n) = \begin{pmatrix} u^n\cdot\xi & \xi_1 & \xi_2 \\ h'(g^n)\xi_1 & (u^n+\epsilon\nu)\cdot\xi & 0 \\ h'(g^n)\xi_2 & 0 & (u^n+\epsilon\nu)\cdot\xi \end{pmatrix},$$

which can be symmetrized by

$$S(g^n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{h'(g^n)} & 0 \\ 0 & 0 & \frac{1}{h'(g^n)} \end{pmatrix}$$

Define $\Omega_\delta = \{x \in \overline{\Omega} \mid \mathrm{dist}(x,\partial\Omega) \le \delta\}$, then we can find an open covering $\{U_j\}_{1\le j\le k}$ of Ω_δ, and a partition of unity $\theta_j(x), j = 1, 2, \cdots k$, and $\theta_0(x)$, such that $\mathrm{supp}\theta_j \subset U_j, \mathrm{supp}\theta_0 \subset \{x \in \Omega \mid \mathrm{dist}(x,\partial\Omega) > \frac{\delta}{2}\}$. Let us assume further that on each U_j, $U_j \cap \partial\Omega$ is given locally as the solution set of

$$0 = \phi_j(x_1,x_2) = x_2 - f_j(x_1), \tag{2.9}$$

such that $U_j \cap \Omega = \{(x_1,x_2) \in U_j \mid \phi_j(x) \ge 0\}$.

Now let us divide the energy estimates into two parts. We first multiply $\theta_0(x)$ to both sides of (2.8) to obtain

$$\partial_t(\theta_0\, w^n) + \sum_{l=1}^{2} A^l(w^n)\,\partial_l(\theta_0\, w^n) = \sum_{l=1}^{2} A^l(w^n)\,\partial_l\,\theta_0\, w^n + \theta_0\, L(w^n). \tag{2.10}$$

Let us denote $v^n = (g^n,\, u^n - \bar{u}(x))$, then by multiplying (2.10) by $2S(g^n)(\theta_0 v^n)$, integrating over the space variables, and notice that $S(g^n)A^l(w^n)$, $l = 1, 2$, are symmetric matrixes, we can use integration by parts to get

$$\begin{aligned} \frac{d}{dt}\int_\Omega (\theta_0\, v^n)\cdot(S(g^n)(\theta_0\, v^n))\,dx &= \int_\Omega \Bigg\{ (\theta_0\, v^n)\cdot((S(g^n))_t\,\theta_0\, v^n) \\ &+ \sum_{l=1}^{2} (\theta_0\, v^n)\cdot\big((S(g^n)A^l(w^n))_{x_l}\,\theta_0\, v^n\big) - \sum_{l=1}^{2}\big(S(g^n)A^l(w^n)\cdot\partial_l(\theta_0\,\bar{u})\big)\cdot(\theta_0\, v^n) \\ &+ 2\sum_{l=1}^{2}\big(A^l(w^n)\,\partial_l\,\theta_0\, w^n\big)\cdot(S(g^n)\,\theta_0\, v^n) + 2(\theta_0\, Lw^n)\cdot(S(g^n)\theta_0 v^n)\Bigg\}\,dx. \end{aligned} \tag{2.11}$$

From (2.11) and the fact that $\theta_0(x) = 1$ for $|x|$ large enough, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (\theta_0\, v^n)\cdot (S(g^n)(\theta_0 v^n))\, dx \\
&\le C\left(\|\partial_t\, g^n\|_{L^\infty}, \|\nabla_x\, w^n\|_{L^\infty},\, \|w^n\|_{L^\infty}\right)\left(1 + \|\theta_0\, v^n\|_{L^2}^2 + \|\nabla_x\,\theta_0\, w^n\|_{L^2}\right) \\
&\le C\left(\|\partial_t\, g^n\|_{L^\infty}, \|\nabla_x v^n\|_{L^\infty}, \|v^n\|_{L^\infty}\right)\left(1 + \|v^n\|_{L^2}^2\right) \qquad (2.12)\\
&\le C(|\!|\!|v^n|\!|\!|_3)(\|v^n\|_{L^2}^2 + 1).
\end{aligned}
$$

On the other hand, let $\alpha = (\alpha_1, \alpha_2)$, then for $1 \le |\alpha| \le 3$, take ∂_x^α to both sides of (2.10), we have

$$
\begin{aligned}
\partial_t\, \partial_x^\alpha(\theta_0 w^n) + \sum_{l=1}^{2} A^l(w^n)\,\partial_l\,\partial_x^\alpha(\theta_0\, w^n) &= \sum_{l=1}^{2}\left[A^l(w^n),\, \partial_x^\alpha\right]\partial_l(\theta_0\, w^n) \\
&+ \sum_{l=1}^{2}\partial_x^\alpha\left(A^l(w^n)\,\partial_l\,\theta_0\, w^n\right) + \partial_x^\alpha(\theta_0\, L(w^n)). \qquad (2.13)
\end{aligned}
$$

We observe that

$$
A^1(w^n) = A(v^n) + \begin{pmatrix} \bar u_1(x) & 0 & 0 \\ 0 & \bar u_1(x) & 0 \\ 0 & 0 & \bar u_1(x) \end{pmatrix}.
$$

One also has a similar observation for $A^2(w^n)$. By the Moser-type calculus inequality, we have

$$
\left\|\left[A^j(v^n),\, \partial_x^\alpha\right]\partial_l\,(\theta_0\, v^n)\right\|_{L^2} \le C\left(\left\|\nabla\, A^j(v^n)\right\|_{L^\infty}\|\theta_0 v^n\|_{H^3} + \|\partial_l\,(\theta_0\, v^n)\|_{L^\infty}\left\|A^j(v^n)\right\|_{H^3}\right).
$$

Thus multiplying (2.13) by $2S(g^n)\,\partial_x^\alpha(\theta_0\, w^n)$, and following the line of estimates of (2.12), we conclude

$$
\frac{d}{dt}\int_\Omega (\partial_x^\alpha(\theta_0 w^n))\cdot (S(g^n)\partial_x^\alpha(\theta_0 w^n))\, dx \le C(|\!|\!|v^n|\!|\!|_3)\sum_{1\le|\alpha|\le 3}\|\partial_x^\alpha w^n\|_{L^2}^2. \qquad (2.14)
$$

Combining (2.12) with (2.14), and by the definition of w^n and v^n, we find

$$
\frac{d}{dt}\sum_{|\alpha|\le 3}\int_\Omega (\partial_x^\alpha(\theta_0 v^n(t,\cdot)))\cdot (S(g^n)(\partial_x^\alpha(\theta_0 v^n(t,\cdot)))\, dx \le C(|\!|\!|v^n(t)|\!|\!|_3), \qquad (2.15)
$$

which together with (2.3)implies that

$$
\frac{d}{dt}\|\theta_0 v^n(t,\cdot)\|_{H^3} \le C(|\!|\!|v^n(t)|\!|\!|_3). \qquad (2.16)
$$

While by applying ∂_t^m to (2.8) for $m = 1, 2, 3$ and manipulate the same energy estimate as that for (2.16), one can obtain

$$
\frac{d}{dt}\,\|\partial_t^m(\theta_0 v^n(t,\cdot))\|_{H^{3-m}} \le C(|\!|\!|v^n(t)|\!|\!|_3). \qquad (2.17)
$$

Combining (2.16) with (2.17), we get

$$\frac{d}{dt}\,|||\theta_0 v^n(t)|||_3 \le C(|||v^n(t)|||_3). \tag{2.18}$$

Next let us turn to the energy estimate near the boundary. On each $U_j \cap \Omega$, by making the standard change of variables: $y_1 = x_1, y_2 = \phi_j(x_1, x_2)$, we are lead to the following equations

$$\begin{cases} \partial_t g^n + u_1^n\,\partial_{y_1} g^n + u^n\cdot\nabla\phi_j\,\partial_{y_2} g^n + \partial_{y_1} u_1^n + \partial_{y_2} u^n\cdot\nabla\phi_j = 0 \\ \partial_t\, u_m^n + (u_1^n + \epsilon\partial_{x_1}\phi_j)\,\partial_{y_1} u_m^n + \big(u^n\cdot\nabla\phi_j + \epsilon|\nabla\phi_j|^2\big)\,\partial_{y_2} u_m^n \\ \quad + \begin{cases} \partial_{y_1} h(g^n) + \partial_{y_2} h(g^n)\,\dfrac{\partial\phi_j}{\partial x_1}, & \text{if } m = 1, \\ \partial_{y_2} h(g^n)\,\dfrac{\partial\phi_j}{\partial x_2}, & \text{if } m = 2, \end{cases} = \epsilon\,(\nabla\phi_j\cdot\nabla_x)\,\tilde{u}_m^n. \end{cases} \tag{2.19}$$

Following the proof of (2.18), one deduces

$$\frac{d}{dt}\left(\sum_{m+\alpha_1\le 3}\int_\Omega \big(\partial_t^m\,\partial_{y_1}^{\alpha_1}\,(\theta_j\,g^n(t,\cdot))\big)\cdot\big(S(g^n)\,\partial_t^m\,\partial_{y_1}^{\alpha_1}\,(\theta_j\,g^n(t,\cdot))\big)\,dx\right) \le C\,(|||v^n(t)|||_3)\,. \tag{2.20}$$

Moreover, the estimate

$$\frac{d}{dt}\left(\sum_{m+|\alpha|\le 2}\int_\Omega (\partial_t^m\partial_y^\alpha(\theta_j g^n(t,\cdot)))\cdot(S(g^n)\partial_t^m\partial_y^\alpha(\theta_j g^n(t,\cdot)))\,dx\right) \le C(|||v^n(t)|||_3) \tag{2.21}$$

can be obtained in the standard fashion by simply integration the spatial derivatives by parts.

On the other hand, from (2.19), we get

$$\begin{pmatrix} u^n\cdot\nabla\phi_j & 1 \\ (1+f_{y_1}^2)\,h'(g^n)\,u^n\cdot\nabla\phi_j + \epsilon|\nabla\phi_j|^2 \end{pmatrix}\begin{pmatrix} \partial_{y_2} g^n \\ \partial_{y_2} u^n\cdot\nabla\phi_j \end{pmatrix} \tag{2.22}$$
$$= -\begin{pmatrix} \partial_t g^n + u_1^n\partial_{y_1} g^n + \partial_{y_1} u_1^n \\ (\partial_t u^n + (u_1^n + \epsilon\partial_{x_1}\phi_j)\,\partial_{y_1} u^n)\cdot\nabla\phi_j + \partial_{y_1} h(g^n)\,\partial_{x_1}\phi_j - \epsilon(\nabla\phi_j\cdot\nabla_x)\,\tilde{u}^n\cdot\nabla_x\phi_j \end{pmatrix}.$$

Since the boundary condition $u^n\cdot\nabla\phi_j\,|_{\partial\Omega} = 0$, we thus have, for $x \in U_j\cap\Omega$, $\bar{x} \in \partial(U_j\cap\Omega)$, that

$$|(u^n\nabla\phi_j)(t,x) - (u^n\cdot\nabla\phi_j)(t,\bar{x})| \le C\|\nabla u^n\|_{L^\infty}\,|x-\bar{x}| \le C\|\nabla u^n(t,\cdot)\|_{H^2}|x-\bar{x}|, \tag{2.23}$$

Thus by taking U_j such that $\mathrm{diam}(\mathrm{supp}\,U_j) \ll 1$, we can solve $(\partial_{y_2} g^n, \partial_{y_2} u^n\cdot\nabla\phi_j)$ through (2.22).

While for any small $\epsilon_2 > 0$, there exists a positive constant $C(\epsilon_2)$ such that there holds (see (4.6) of [Sch86]):

$$\left\|\partial_{y_1}^{k_1} \partial_{y_2}^{k_2} f\right\|_{L^2} \leq \epsilon_2 \left\|\partial_{y_2}^{k_1+k_2} f\right\|_{L^2} + C(\epsilon_2) \left(\sum_{\alpha_1 \leq k_1+k_2} \|\partial_{y_1}^{\alpha_1} f\|_{L^2} + \|f\|_{H^{k_1+k_2-1}} \right). \tag{2.24}$$

By combining (2.22-2.24) we get

$$\begin{aligned} |||(\theta_j\, \partial_{y_2} g^n, \theta_j\, \partial_{y_2} u^n \cdot \nabla\phi_j|||_2 \leq C(|||v^n|||_3) \Bigg(C(\epsilon_2) \Bigg(1 + \sum_{m+\alpha_1 \leq 3} \|\partial_t^m \partial_{y_1}^{\alpha_1} v^n\|_{L^2} \Bigg) \\ + \epsilon_2 \left(|||(\theta_j\, \partial_{y_2} g^n, \theta_j\, \partial_{y_2} u^n \cdot \nabla\phi_j)|||_2 + |||\theta_j\, \partial_{y_2} u_1^n|||_2 \right) \Bigg). \end{aligned} \tag{2.25}$$

On the other hand, by the special structure of the equation in (2.1), and a easy calculation as that in [Sch86], we have

$$\frac{d}{dt} |||\nabla \times u^n|||_2 \leq C(|||(g^n(t), u^n(t))|||_3. \tag{2.26}$$

Since $\partial_{x_1} \phi_j(x) = -f_j'(x_1)$, $\partial_{x_2} \phi_j(x) = 1$, and since

$$\begin{pmatrix} -f_j' & 1 \\ 1 & f_j' \end{pmatrix} \begin{pmatrix} \partial_{y_2} u_1^n \\ \partial_{y_2} u_2^n \end{pmatrix} = \begin{pmatrix} \partial_{y_2} u^n \cdot \nabla\phi_j \\ -\partial_{y_1} u_2^n + \nabla \times u^n \end{pmatrix}, \tag{2.27}$$

we hence conclude

$$|||\partial_{y_2} u^n|||_2 \leq K \left(|||\partial_{y_2} u^n \cdot \nabla\phi_j|||_3 + |||\partial_{y_1} u_2^n|||_2 + |||\nabla \times u^n|||_2 \right). \tag{2.28}$$

Iterating estimates (2.27) and (2.28) several times, we lead to

$$|||\partial_{y_2} u^n|||_2 \leq C \left(\sum_{m+\alpha_1 \leq 3} \|\partial_t^m \partial_{y_1}^{\alpha_1} w^n\|_{L^2} + |||v^n|||_2 + |||\nabla \times u^n|||_2 \right). \tag{2.29}$$

Substituting (2.29) to (2.25), we get for ϵ_2 small enough, there holds

$$|||\theta_j\, \partial_{y_2} w^n|||_2 \leq C(|||v^n|||_3) \left(\sum_{m+\alpha_1 \leq 3} \|\partial_t^m \partial_{y_1}^{\alpha_1} v^n\|_{L^2} + |||v^n|||_2 + |||\nabla \times u^n|||_2 \right) \tag{2.30}$$

From (2.3), (2.20) and (2.30), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\sum_{m+\alpha_1 \leq 3} \|\partial_t^m \partial_{y_1}^{\alpha_1} (\theta_j\, w^n(t,\cdot))\, \|_{L^2} + |||\, (\theta_j\, g^n(t,\cdot), \theta_j\, v^n(t,\cdot), \nabla \times v^n)\, |||_2 \right) \\ \leq C_j \left(\sum_{m+\alpha_1 \leq 3} \|\partial_t^m \partial_{y_1}^{\alpha_1} g^n(t,\cdot)\|_{L^2}, |||(\theta_j\, g^n(t,\cdot), \theta_j\, v^n(t,\cdot), \nabla \times v^n)|||_2 \right). \end{aligned} \tag{2.31}$$

By summing up estimate (2.31) for j from 1 to k together with (2.18), we conclude, for a positive constant T^*, and for any $T < T^*$, that

$$\sum_{j=1}^{k}\left(\sum_{m+\alpha_1\leq 3}\|\partial_t^m\,\partial_{y_1}^{\alpha_1}\left(\theta_j\,w^n(t,\cdot)\right)\|_{L^2}+\|\|\left(\theta_j\,g^n(t,\cdot),\theta_j\,v^n(t,\cdot),\nabla\times v^n\right)\|\|_2\right)$$
$$+\,\|\|\theta_0\,v^n(t)\|\|_3\leq C\left(T,\Omega,\|(g_0,v_0)\|_{H^3}\right). \tag{2.32}$$

Substituting (2.32) to (2.30), we obtain the estimate

$$\|\|(g^n(t,\cdot),\,v^n(t,\cdot))\|\|_{3,T}\leq C\left(T,\Omega,\|(g_0,v_0)\|_{H^3}\right). \tag{2.33}$$

Thus if we take $\epsilon=\min\left(\tilde{\epsilon}(n),\frac{1}{n}\right)$, as [Sch86], it is standard to prove the convergence of $v^n(t,x)$ in $L^\infty\left([0,T],L^2(\Omega)\right)$ to $v(t,x)=(g(t,x),u(t,x)-\bar{u}(x))\in X_{3,T}$. This procedure also implies the uniqueness of smooth solution to the equation (2.4). This completes the proof of the Theorem. $\square$

3 The proof of Theorem 1.1

In this section, we will employ and improve some arguments in [ZP1] and [ZP2] to prove Theorem 1.1. If the initial data of (1.1) satisfies (A1) in the introduction, by Theorem 4.1 in the Appendix, we know that (1.1) has a unique global smooth solution $\psi^\epsilon(t,x)$ such that $\partial_t^j\,\partial_x^\alpha\left(\psi^\epsilon(t,x)-A^\epsilon(t,x)\right)\in L^\infty([0,T],H^{s-2j-|\alpha|}(\Omega))$ for all $T<\infty, 1\leq 2j+|\alpha|\leq 3$, where $A^\epsilon(t,x)=\chi(x)\,e^{\frac{i}{\epsilon}\left(u^\infty\cdot x-\frac{|u^\infty|^2}{2}\,t\right)}$, and $\chi(x)\in C^\infty(\mathbb{R}^2)$ with $\chi(x)=\begin{cases}0, & \text{for } |x|\leq R\\ 1, & \text{for } |x|\geq 2R,\end{cases}$ and R is big enough such that $\Omega^c\subset B_R(0)$.

Before we proceed further, let us first modify the madelung's fluid dynamic equation to the following form, see [LX].

Lemma 3.1. *Let $\rho^\epsilon(t,x)=:|\psi^\epsilon(t,x)|^2$, $J_j^\epsilon(t,x)=:\epsilon Im(\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon)$. Then there holds*

1)

$$\partial_t\,\rho^\epsilon+div\,J^\epsilon=0, \tag{3.1}$$

$$\partial_t\,J_j^\epsilon+\frac{\epsilon^2}{4}\sum_{k=1}^{2}\partial_k\left(4\Re(\partial_j\,\psi^\epsilon\,\partial_k\,\overline{\psi^\epsilon})-\partial_j\,\partial_k|\psi^\epsilon|^2\right)+\frac{1}{2}\,\partial_j(\rho^\epsilon)^2=0. \tag{3.2}$$

2) Let R be large enough such that $|x+u^\infty T^*|\leq R$ for all $x\in\partial\Omega$, then for $0\leq t\leq T^*$, there holds

$$\int_\Omega\left(\epsilon^2(1-\chi)|\nabla\psi^\epsilon|^2+\chi|\epsilon\nabla\psi^\epsilon-iu^\infty\,\psi^\epsilon|^2\right)dx+\int_\Omega\left(\rho^\epsilon(t,x)-1\right)^2\,dx\leq Ce^{Ct}, \tag{3.3}$$

where C is a constant depending only on $\chi(x)$ and various constants in the assumptions (A1)–(A3) in the introduction.

Proof. 1) Multiplying (1.1) by $\overline{\psi^\epsilon}$, we get

$$i\epsilon\,\partial_t\,\psi^\epsilon\overline{\psi^\epsilon} = -\frac{\epsilon^2}{2}\,\mathrm{div}\left(\nabla\psi^\epsilon\overline{\psi^\epsilon}\right) + \frac{\epsilon^2}{2}\,|\nabla\psi^\epsilon|^2 + \left(|\psi^\epsilon|^2-1\right)|\psi^\epsilon|^2.$$

Take the imaginary part of the above equation, we get (3.1).

Next we multiply the conjugate equation of (1.1) by $\partial_j\,\psi^\epsilon$, to obtain

$$i\epsilon\,\partial_t\,\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon = \frac{\epsilon^2}{2}\sum_{k=1}^{2}\partial_k\left(\partial_k\,\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon\right) - \frac{\epsilon^2}{2}\sum_{k=1}^{2}\partial_k\,\overline{\psi^\epsilon}\,\partial_j\,\partial_k\,\psi^\epsilon - (|\psi^\epsilon|^2-1)\,\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon. \tag{3.4}$$

While by applying ∂_j to the first equation of (1.1), and multiplying the resulting equation by $\overline{\psi^\epsilon}$, we get

$$\begin{aligned} i\epsilon\,\partial_t\,\partial_j\,\psi^\epsilon\,\overline{\psi^\epsilon} &= -\frac{\epsilon^2}{2}\,\Delta\,\partial_j\,\psi^\epsilon\,\overline{\psi^\epsilon} + \partial_j(|\psi^\epsilon|^2)|\psi^\epsilon|^2 + (|\psi^\epsilon|^2-1)\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon \qquad (3.5)\\ &= -\frac{\epsilon^2}{2}\sum_{k=1}^{2}\partial_k(\partial_j\,\partial_k\,\psi^\epsilon\,\overline{\psi^\epsilon}) + \frac{\epsilon^2}{2}\sum_{k=1}^{2}\partial_j\,\partial_k\,\psi^\epsilon\,\partial_k\,\overline{\psi^\epsilon} + \frac{1}{2}\,\partial_j(|\psi^\epsilon|^4) + (|\psi^\epsilon|^2-1)\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon. \end{aligned}$$

Summing up (3.4) and (3.5), we find

$$i\epsilon\,\partial_t\left(\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon\right) = \frac{\epsilon^2}{2}\sum_{k=1}^{2}\partial_k\left(\partial_k\,\overline{\psi^\epsilon}\,\partial_j\,\psi^\epsilon - \partial_j\,\partial_k\,\psi^\epsilon\,\overline{\psi^\epsilon}\right) + \frac{1}{2}\,\partial_j(\rho^\epsilon)^2.$$

To get (3.2), we simply take the real part of the above equation.

2) By taking the complex L^2 inner product of (1.1) with $(1-\chi(x))\partial_t\psi^\epsilon$, and taking the real part, we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\epsilon^2}{2}\int_\Omega(1-\chi)|\nabla\psi^\epsilon|^2\,dx + \frac{1}{2}\int_\Omega(1-\chi)(|\psi^\epsilon|^2-1)^2\,dx\right\}\\ &\qquad = -\epsilon^2\,\Re\int_\Omega \nabla\psi^\epsilon\,\nabla(1-\chi)\,\partial_t\overline{\psi^\epsilon}\,dx. \end{aligned} \tag{3.6}$$

In order to prove (3.3), we need to estimate the other terms in the left hand side of (3.3). For this purpose, we denote

$$\phi^\epsilon(t,x) = \psi^\epsilon(t,x+u^\infty t)\,e^{-\frac{i}{\epsilon}\left(u^\infty\cdot x+\frac{|u^\infty|^2}{2}t\right)}, \tag{3.7}$$

then by a trivial calculation, we find that $\phi^\epsilon(t,x)$ satisfies:

$$i\epsilon\,\partial_t\,\phi^\epsilon = -\frac{\epsilon^2}{2}\,\Delta\,\phi^\epsilon + \left(|\phi^\epsilon|^2-1\right)\phi^\epsilon. \tag{3.8}$$

Note that by the definition of ϕ^ϵ and χ, $\chi(x+u^\infty t)\,\frac{\partial\phi^\epsilon}{\partial\nu}\mid_{\partial\Omega}=0$. By taking the complex L^2 inner product of (3.8) with $\chi(x+u^\infty t)\,\partial_t\,\phi^\epsilon$, and by taking the real part, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\epsilon^2}{2}\int_\Omega \chi(x+u^\infty\, t)|\nabla\phi^\epsilon|^2\,dx+\frac{1}{2}\int_\Omega \chi(x+u^\infty\, t)(|\phi^\epsilon|^2-1)^2\,dx\right\}\\
&\qquad\qquad = -\epsilon^2\,\Re\int_\Omega \nabla\phi^\epsilon\,\nabla\chi(x+u^\infty\, t)\,\partial_t\,\overline{\phi^\epsilon}\,dx\\
&\qquad\qquad +\frac{1}{2}\int_\Omega u^\infty\cdot\nabla\chi(x+u^\infty\, t)\left(\epsilon^2|\nabla\phi^\epsilon|^2+(|\phi^\epsilon|^2-1)^2\right)dx. \qquad (3.9)
\end{aligned}$$

Substituting (3.7) into (3.9), and using the change of variables that $y=x+u^\infty t$ in the above integral, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\int_\Omega \chi\left(|\epsilon\nabla\psi^\epsilon-iu^\infty\,\psi^\epsilon|^2+(|\psi^\epsilon|^2-1)^2\right)dx\\
&=\frac{1}{2}\int_\Omega u^\infty\cdot\nabla\chi|\epsilon\nabla\,\psi^\epsilon-iu^\infty\,\psi^\epsilon|^2\,dx+\frac{1}{2}\int_\Omega u^\infty\cdot\nabla\chi\left(|\psi^\epsilon|^2-1\right)^2\,dx\\
&+\epsilon\,\Re\int_\Omega (iu^\infty\,\psi^\epsilon-\epsilon\nabla\,\psi^\epsilon)\,\nabla\chi\left(u^\infty\,\nabla\overline{\psi^\epsilon}+\frac{i}{2\epsilon}\,|u^\infty|^2\overline{\psi^\epsilon}\right)dx\\
&-\epsilon^2\,\Re\int_\Omega \nabla\psi^\epsilon\,\nabla\chi\,\partial_t\,\overline{\psi^\epsilon}\,dx+\epsilon\,\Re\int_\Omega\left(iu^\infty\cdot\nabla\chi\,\psi^\epsilon\,\partial_t\,\overline{\psi^\epsilon}\right)dx. \qquad (3.10)
\end{aligned}$$

One notices that the second line of (3.6) plus the first term in the last line of (3.10) equal 0, by adding up (3.6) and (3.10), we find

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\int_\Omega\left(\epsilon^2(1-\chi)\,|\nabla\psi^\epsilon|^2+\chi\,|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2+\left(|\psi^\epsilon|^2-1\right)^2\right)dx\\
&=\frac{1}{2}\int_\Omega u^\infty\nabla\chi|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2\,dx+\frac{1}{2}\int_\Omega u^\infty\nabla\chi(|\psi^\epsilon|^2-1)^2\,dx\\
&+\epsilon\Re\int_\Omega (iu^\infty\psi^\epsilon-\epsilon\nabla\psi^\epsilon)\,\nabla\chi\left(u^\infty\nabla\overline{\psi^\epsilon}+\frac{i}{2\epsilon}|u^\infty|^2\overline{\psi^\epsilon}\right)dx\\
&+\epsilon\Re\int_\Omega iu^\infty\cdot\nabla\chi\psi^\epsilon\,\partial_t\overline{\psi^\epsilon}\,dx. \qquad (3.11)
\end{aligned}$$

To complete the proof of (3.3), we shall estimate the right-hand side of the above formula term by term. First of all, since $\nabla\chi$ has compact support, we have

$$\begin{aligned}
&\int_\Omega u^\infty\cdot\nabla\chi|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2\,dx=\int_\Omega u^\infty\cdot\nabla\chi\left(\chi+(1-\chi)\right)|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2\,dx\\
&\le C\left\{\int_\Omega \chi|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2\,dx+\int_\Omega(1-\chi)\left(\epsilon^2|\nabla\psi^\epsilon|^2+|\psi^\epsilon|^2\right)dx\right)\\
&\le C\left\{1+\int_\Omega\left((1-\chi)|\nabla\psi^\epsilon|^2+\chi\,|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2+\left(|\psi^\epsilon|^2-1\right)^2\right)dx\right\}, \qquad (3.12)
\end{aligned}$$

Similarly, we can estimate the second term on the right-hand side of (3.11) as follows:

$$
\begin{aligned}
&\left| \epsilon \Re \int_{\Omega} (iu^\infty \psi^\epsilon - \epsilon \nabla \psi^\epsilon)\, \nabla \chi \left(u^\infty \nabla \overline{\psi^\epsilon} + \frac{i}{2\epsilon} \left| u^\infty \right|^2 \overline{\psi^\epsilon} \right) dx \right| \\
&\leq C \left\{ 1 + \int_{\Omega} \left((1-\chi) |\nabla \psi^\epsilon|^2 + \chi \left| \epsilon \nabla \psi^\epsilon - iu^\infty \psi^\epsilon \right|^2 + \left(|\psi^\epsilon|^2 - 1 \right)^2 \right) dx \right\}. \qquad (3.13)
\end{aligned}
$$

While by using the first equation of (1.1), we have

$$
\begin{aligned}
\epsilon \Re \int_{\Omega} iu^\infty\, \nabla \chi\, \psi^\epsilon\, \partial_t \overline{\psi^\epsilon}\, dx = &\frac{\epsilon^2}{4} \int_{\Omega} \Delta \left(u^\infty \cdot \nabla \chi \right) \rho^\epsilon \, dx - \frac{\epsilon^2}{2} \int_{\Omega} u^\infty \cdot \nabla \chi |\nabla \psi^\epsilon|^2 \, dx \\
&- \int_{\Omega} u^\infty \cdot \nabla \chi (\rho^\epsilon - 1)\, \rho^\epsilon \, dx,
\end{aligned} \qquad (3.14)
$$

then we follow the same line of estimates as for (3.12) to handle $\epsilon \Re \int_\Omega iu^\infty \nabla \chi \psi^\epsilon\, \partial_t \overline{\psi^\epsilon}\, dx$.

By combining (3.6)–(3.14) all together, we find

$$
\begin{aligned}
&\frac{d}{dt} \int_{\Omega} \left(\epsilon^2 (1-\chi)\, |\nabla \psi^\epsilon|^2 + \chi |\epsilon \nabla \psi^\epsilon - iu^\infty\, \psi^\epsilon|^2 + \left(|\psi^\epsilon|^2 - 1 \right)^2 \right) dx \\
&\leq C \left\{ 1 + \int_{\Omega} \left(\epsilon^2 (1-\chi)\, |\nabla \psi^\epsilon|^2 + \chi |\epsilon \nabla \psi^\epsilon - iu^\infty \psi^\epsilon|^2 + \left(|\psi^\epsilon|^2 - 1 \right)^2 \right) dx \right\}.
\end{aligned} \qquad (3.15)
$$

Note that by the assumptions (A1)–(A3) in the introduction, we have

$$
\begin{aligned}
\int_{\Omega} |\epsilon \nabla \psi_0^\epsilon - iu^\infty \psi^\epsilon|^2 \, dx \leq &2 \int_{\Omega} \left| \epsilon \nabla \left(\psi_0^\epsilon - e^{\frac{iu^\infty x}{\epsilon}} \right) \right|^2 dx \; + 2|u^\infty|^2 \int_{\Omega} \left| \psi_0^\epsilon - e^{\frac{iu^\infty x}{\epsilon}} \right|^2 dx \\
&\leq C,
\end{aligned} \qquad (3.16)
$$

(3.15), (3.16) together with the Gronwall inequality yield (3.3). This completes the proof of the Lemma. ☐

Remark 3.1. Notice by the argument at the beginning of this section, we know that $\psi^\epsilon(t,x) - A^\epsilon(t,x) \to 0$ as $|x| \to \infty$. However, we do not know how to obtain the uniform estimate for $\int_\Omega |\epsilon \nabla (\psi^\epsilon - A^\epsilon)|^2 \, dx + \int_\Omega (\rho^\epsilon - 1)^2 \, dx$, as we did in the Appendix for ϵ fixed case.

Let $(\rho_0(x), u_0(x))$ satisfies (A3) in the introduction, then by Theorem 2.1, (1.9–1.10) has a unique local smooth solution $(\rho(t,x), u(t,x))$ with $(\rho(t,x) - 1,\, u(t,x) - u^\infty) \in X_{3,T}$, for any $T < T^*$. As it was pointed out in the introduction, We shall study the evolution to the following functional:

$$
H^\epsilon(t) =: \frac{1}{2} \int_{\Omega} |(\epsilon \nabla_x - iu)\psi^\epsilon|^2 \, dx + \frac{1}{2} \int_{\Omega} |\rho^\epsilon - \rho|^2 \, dx, \qquad (3.17)
$$

for $0 < t < T^*$.

The key ingredient in the proof of Theorem 1.1 will then be the following lemma:

Lemma 3.2. *Let $H^\epsilon(t)$ be defined as (3.17). Then we have*

$$\frac{d}{dt}\,H^\epsilon(t) = -\sum_{j,k=1}^{2}\int_\Omega \partial_j\, u_k Re\left(\left(\epsilon\partial_{x_j} - iu_j\right)\psi^\epsilon\overline{\left(\epsilon\partial_{x_k} - iu_k\right)\psi^\epsilon}\right)dx$$
$$-\frac{1}{2}\int_\Omega div\,u\,(\rho^\epsilon-\rho)^2\,dx + \frac{\epsilon^2}{4}\int_\Omega \nabla\rho^\epsilon(\nabla\,div\,u)\,dx. \tag{3.18}$$

Proof. Let χ be the cutoff function defined in 2) of Lemma 3.1. From the definition of $H^\epsilon(t)$, we can rewrite $H^\epsilon(t)$ in the following way:

$$H^\epsilon(t) = \frac{1}{2}\int_\Omega\left(\epsilon^2(1-\chi)\,|\nabla_x\psi^\epsilon|^2 + \chi|\epsilon\nabla_x\psi^\epsilon - iu^\infty\psi^\epsilon|^2 + (\rho^\epsilon-1)^2\right)dx$$
$$-\int_\Omega(u-\chi u^\infty)\cdot(J^\epsilon - u^\infty\rho^\epsilon)\,dx + \frac{1}{2}\int_\Omega\rho^\epsilon\left(|u|^2 - 2u\cdot u^\infty + \chi|u^\infty|^2\right)dx$$
$$+\frac{1}{2}\int_\Omega\left((\rho-1)^2 - 2(\rho-1)(\rho^\epsilon-1)\right)dx. \tag{3.19}$$

We now calculate the time derivative of the above expression term by term.

By (3.2), we obtain first that

$$\begin{aligned}
&-\frac{d}{dt}\int_\Omega(u-\chi u^\infty)\cdot(J^\epsilon-u^\infty\rho^\epsilon)\,dx\\
&=\int_\Omega\left(-(u-\chi u^\infty)\cdot(\partial_t J^\epsilon - u^\infty\partial_t\rho^\epsilon) - \partial_t u\cdot(J^\epsilon-u^\infty\rho^\epsilon)\right)dx\\
&=\int_\Omega\sum_{j=1}^{2}\left(\frac{\epsilon^2}{4}\sum_{k=1}^{2}\partial_k\left(4\Re(\partial_j\psi^\epsilon\overline{\partial_k\psi^\epsilon}) - \partial_j\partial_k|\psi^\epsilon|^2\right) - u^\infty\,div\,J^\epsilon\right)\left(u_j-\chi u_j^\infty\right)dx\\
&+\frac{1}{2}\int_\Omega\nabla(\rho^\epsilon)^2\cdot(u-\chi u^\infty)\,dx - \int_\Omega\partial_t u\cdot(J^\epsilon-u^\infty\rho^\epsilon)\,dx.
\end{aligned}\tag{3.20}$$

Using the boundary conditions in (1.8),(1.6) and (1.1) that $u\cdot\nu\mid_{\partial\Omega}=0$, $\frac{\partial\psi^\epsilon}{\partial\nu}\mid_{\partial\Omega}=0$, and by integration by parts, we have the followings

$$\int_\Omega\nabla(\rho^\epsilon)^2\cdot(u-\chi u^\infty)\,dx = \int_{\partial\Omega}(\rho^\epsilon)^2u\cdot\nu\,dS - \int_\Omega\left((\rho^\epsilon)^2-1\right)\mathrm{div}\,(u-\chi u^\infty)\,dx$$
$$= -\int_\Omega\left((\rho^\epsilon)^2-1\right)\mathrm{div}\,(u-\chi u^\infty)\,dx, \tag{3.21}$$

$$
\begin{aligned}
&\sum_{j=1}^{2}\int_{\Omega}\Delta\,\partial_j\left(|\psi^\epsilon|^2\right)\cdot\left(u_j-\chi u_j^\infty\right)dx\\
&=\int_{\partial\Omega}\Delta|\psi^\epsilon|^2\,u\cdot\nu\,dS-\int_{\Omega}\Delta\rho^\epsilon\mathrm{div}\,(u-\chi u^\infty)\,dx\\
&=-\int_{\Omega}\Delta\rho^\epsilon\mathrm{div}\,(u-\chi u^\infty)\,dx\\
&=-\int_{\partial\Omega}\frac{\partial\rho^\epsilon}{\partial\nu}\mathrm{div}\,(u-\chi u^\infty)\,dS+\int_{\Omega}\nabla\rho^\epsilon\cdot\nabla\mathrm{div}\,(u-\chi u^\infty)\,dx\\
&=\int_{\Omega}\nabla\rho^\epsilon\cdot\nabla\mathrm{div}\,(u-\chi u^\infty)\,dx.
\end{aligned}
\tag{3.22}
$$

Here dS is the surface measure on $\partial\Omega$. We also note that

$$
\begin{aligned}
&\int_{\Omega}\sum_{j=1}^{2}\left(\epsilon^2\sum_{k=1}^{2}\partial_k\ \mathrm{Re}\left(\partial_j\psi^\epsilon\,\partial_k\overline{\psi^\epsilon}\right)-u_j^\infty\mathrm{div}J^\epsilon\right)\left(u_j-\chi u_j^\infty\right)dx\\
&=\int_{\partial\Omega}\sum_{j=1}^{2}\left(\epsilon^2\mathrm{Re}\left(\partial_j\psi^\epsilon\overline{\frac{\partial\psi^\epsilon}{\partial\nu}}\right)+u_j^\infty J^\epsilon\cdot\nu\right)\left(u_j-\chi u_j^\infty\right)dS\\
&-\int_{\Omega}\sum_{j,k=1}^{2}\left(\epsilon^2\ \mathrm{Re}\left(\partial_j\psi^\epsilon\,\partial_k\overline{\psi^\epsilon}\right)-u_j^\infty J_k^\epsilon\right)\partial_k\left(u_j-\chi u^\infty\right)dx\\
&=-\int_{\Omega}\sum_{j,k=1}^{2}\left(\epsilon^2\ \mathrm{Re}\left(\partial_j\psi^\epsilon\,\partial_k\overline{\psi^\epsilon}\right)-u_j^\infty J_k^\epsilon\right)\partial_k\left(u_j-\chi u^\infty\right)dx.
\end{aligned}
\tag{3.23}
$$

Combining (3.20) with (3.23), we get

$$
\begin{aligned}
&-\frac{d}{dt}\int_{\Omega}(u-\chi u^\infty)\cdot(J^\epsilon-\rho^\epsilon u^\infty)\,dx\\
&=-\frac{\epsilon^2}{4}\int_{\Omega}\nabla\rho^\epsilon\cdot\nabla\mathrm{div}\,(u-\chi u^\infty)\,dx-\frac{1}{2}\int_{\Omega}\left((\rho^\epsilon)^2-1\right)\mathrm{div}\,(u-\chi u^\infty)\,dx\\
&-\int_{\Omega}\sum_{j,k=1}^{2}\left(\epsilon^2\ \mathrm{Re}\left(\partial_j\psi^\epsilon\,\partial_k\overline{\psi^\epsilon}\right)-u_j^\infty J_k^\epsilon\right)\partial_k\left(u_j-\chi u_j^\infty\right)dx\\
&-\int_{\Omega}(J^\epsilon-u^\infty\rho^\epsilon)\cdot\partial_t u\,dx.
\end{aligned}
\tag{3.24}
$$

Next we observe that the boundary condition $\frac{\partial\psi^\epsilon}{\partial\nu}\mid_{\partial\Omega}=0$ implies $J^\epsilon\cdot\nu\mid_{\partial\Omega}=0$. Thus by (3.1) and integration by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^\epsilon\left(|u|^2-2u\cdot u^\infty+\chi|u^\infty|^2\right)dx\\
&=\frac{1}{2}\int_\Omega \partial_t\rho^\epsilon\left(|u|^2-2u\cdot u^\infty+\chi|u^\infty|^2\right)dx+\int_\Omega \rho^\epsilon(u-u^\infty)\cdot\partial_t u\,dx\\
&=-\frac{1}{2}\int_\Omega \mathrm{div}J^\epsilon\left(|u|^2-2u\cdot u^\infty+\chi|u^\infty|^2\right)dx+\int_\Omega \rho^\epsilon(u-u^\infty)\cdot\partial_t u\,dx\\
&=\frac{1}{2}\int_\Omega J^\epsilon\cdot\nabla\left(|u|^2-2u\cdot u^\infty+\chi|u^\infty|^2\right)dx+\int_\Omega \rho^\epsilon(u-u^\infty)\cdot\partial_t u\,dx.
\end{aligned}
\tag{3.25}
$$

On the other hand, from the limit equation (1.7), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho-1)^2\,dx&=\int_\Omega(\rho-1)\,\partial_t\rho\,dx=-\int_\Omega(\rho-1)\,\mathrm{div}\,(\rho u)\,dx\\
&=-\frac{1}{2}\int_\Omega(\rho^2-1)\mathrm{div}u\,dx.
\end{aligned}
\tag{3.26}
$$

Finally, by (3.1) and $J^\epsilon\cdot\nu\,|_{\partial\Omega}=0$ again, we can calculate

$$
\begin{aligned}
&-\frac{d}{dt}\int_\Omega(\rho-1)(\rho^\epsilon-1)\,dx=-\int_\Omega\left(\partial_t\,\rho(\rho^\epsilon-1)+(\rho-1)\,\partial_t\,\rho^\epsilon\right)dx\\
&=\int_\Omega\left(\mathrm{div}(\rho u)(\rho^\epsilon-1)+(\rho-1)\,\mathrm{div}\,(\chi u^\infty(\rho^\epsilon-1))+(\rho-1)\,\mathrm{div}\,(\chi u^\infty)\right.\\
&\left.-\nabla\rho\,(J^\epsilon-\chi u^\infty\rho^\epsilon)\right)dx\\
&=\int_\Omega\left(\mathrm{div}(\rho u)(\rho^\epsilon-1)-u^\infty\cdot\nabla\rho\chi(\rho^\epsilon-1)+(\rho-1)\,u^\infty\cdot\nabla\chi-\nabla\rho\,(J^\epsilon-\chi u^\infty\rho^\epsilon)\right)dx\\
&=\int_\Omega\left(\rho\rho^\epsilon\mathrm{div}u+\rho^\epsilon u\cdot\nabla\rho-\nabla\rho\cdot J^\epsilon\right)dx.
\end{aligned}
\tag{3.27}
$$

We also observe, by (3.11) and (3.14), that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\epsilon^2(1-\chi)|\,\nabla\psi^\epsilon|^2+\chi|\epsilon\nabla\psi^\epsilon-iu^\infty\psi^\epsilon|^2+\left(|\psi^\epsilon|^2-1\right)^2\right)dx\\
&=\frac{\epsilon^2}{4}\int_\Omega\Delta\left(u^\infty\nabla\chi\right)\rho^\epsilon\,dx+\int_\Omega\left(\frac{1}{2}\sum_{k=1}^2|u^\infty|^2\;J_k^\epsilon\,\partial_k\chi-\epsilon^2\sum_{j,k=1}^2\Re\left(\partial_j\psi^\epsilon\,\partial_k\overline{\psi^\epsilon}\right)u_k^\infty\partial_j\chi\right.\\
&\left.-\frac{1}{2}\,u^\infty\cdot\nabla\chi(\rho^\epsilon)^2\right)dx.
\end{aligned}
\tag{3.28}
$$

By combining (3.19) and (3.24-3.28), we conclude

$$
\begin{aligned}
\frac{d}{dt}\, H^\epsilon(t) = &\int_\Omega \Bigg(-\epsilon^2 \sum_{j,k=1}^{2} \int_\Omega \mathrm{Re}\left(\partial_j \psi^\epsilon\, \partial_k \overline{\psi^\epsilon}\right) \partial_k u_j - J^\epsilon \cdot (\partial_t u + \nabla\rho) \\
&+\rho^\epsilon u \cdot (\partial_t u + \nabla \rho) + \frac{1}{2}\, J^\epsilon \cdot \nabla |u|^2 \Bigg)\, dx \\
&- \frac{1}{2} \int_\Omega \mathrm{div}\, u \left((\rho^\epsilon)^2 + \rho^2 - 2\,\rho\rho^\epsilon \right) dx + \frac{\epsilon^2}{4} \int_\Omega \nabla \rho^\epsilon \cdot \nabla \mathrm{div}\, u\, dx \\
= &\int_\Omega \Bigg(-\epsilon^2 \sum_{j,k=1}^{2} \mathrm{Re}\left(\partial_j \psi^\epsilon\, \partial_k \overline{\psi^\epsilon}\right) \partial_k u_j + J^\epsilon \cdot (u\cdot\nabla u) - \rho^\epsilon u \cdot (u \cdot \nabla u) \\
&+ \frac{1}{2}\, J^\epsilon \cdot \nabla |u|^2 \Bigg)\, dx - \frac{1}{2} \int_\Omega \mathrm{div}\, u\, (\rho^\epsilon - \rho)^2\, dx + \frac{\epsilon^2}{4} \int_\Omega \nabla\rho^\epsilon \cdot \nabla \mathrm{div} u\, dx \\
= &- \sum_{j,k=1}^{2} \int_\Omega \partial_k u_j \ \mathrm{Re}\left((\epsilon \partial_{x_j} - i u_j)\, \psi^\epsilon\, \overline{(\epsilon\partial_{x_k} - i u_k)\psi^\epsilon} \right) dx \\
&- \frac{1}{2} \int_\Omega \mathrm{div} u\, (\rho^\epsilon - \rho)^2\, dx + \frac{\epsilon^2}{4} \int_\Omega \nabla \rho^\epsilon \cdot \nabla \mathrm{div}\, u\, dx
\end{aligned}
\tag{3.29}
$$

This completes the proof of the Lemma. □

Proof of Theorem 1.1. First of all we have, by (3.3), that

$$
\begin{aligned}
&\epsilon^2 \left| \int_\Omega \nabla\rho^\epsilon \cdot (\nabla \mathrm{div}\, u)\, dx \right| \\
&= \epsilon \left| \int_\Omega \left(\psi^\epsilon\, \overline{(\epsilon\nabla\psi^\epsilon - i u^\infty \psi^\epsilon)} + (\epsilon \nabla \psi^\epsilon - i u^\infty \psi^\epsilon)\, \overline{\psi^\epsilon} \right) \nabla \mathrm{div}\, u\, dx \right| \\
&\le 2\epsilon \left(\int_\Omega |\epsilon \nabla \psi^\epsilon - i u^\infty \psi^\epsilon|^2\, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(|\psi^\epsilon|^2 - 1 \right) |\nabla \mathrm{div}\, u|^2\, dx + \int_\Omega |\nabla \mathrm{div}\, u|^2\, dx \right)^{\frac{1}{2}} \\
&\le C\epsilon \left(\left(\int_\Omega \left(|\psi^\epsilon|^2 - 1 \right)^2\, dx \right)^{\frac{1}{4}} \left(\int_\Omega |\nabla \mathrm{div} u|^4\, dx \right)^{\frac{1}{4}} + \|\mathrm{div}\, u\|_{H^1} \right) \\
&\le C\epsilon \|\nabla u(t,\cdot)\|_{H^2(\Omega)}.
\end{aligned}
\tag{3.30}
$$

Next, from (3.18), (3.30) and the Gronwall inequality, we obtain

$$
H^\epsilon(t) \le C(T)\, e^{\int_0^t \|\nabla\, u(s,\cdot)\|_{L^\infty}\, ds} \left(H^\epsilon(0) + \epsilon \right), \qquad 0 < t \le T < T^*. \tag{3.31}
$$

Finally by assumption (A2) in the introduction, we conclude

$$\begin{aligned} H^\epsilon(0) &= \frac{1}{2}\int_\Omega \left|\left(\epsilon\nabla_x - iu_0\right)\psi_0^\epsilon\right|^2\,dx + \frac{1}{2}\int_\Omega \left(\rho_0^\epsilon - \rho_0\right)^2\,dx \\ &\le \frac{1}{2}\int_\Omega \rho_0^\epsilon\left|u_0 - \nabla S^\epsilon\right|^2\,dx + \epsilon\int_\Omega \left|\nabla\sqrt{\rho_0^\epsilon}\right|^2\,dx \\ &+ \frac{1}{2}\int_\Omega \left(\rho_0^\epsilon - \rho\right)^2\,dx = o(1), \qquad \text{as } \epsilon\to 0. \end{aligned} \tag{3.32}$$

Therefore, one has

$$\lim_{\epsilon\to 0} H^\epsilon(t) = 0, \quad 0\le t\le T < T^*. \tag{3.33}$$

In particular, (3.33) implies that

$$\rho^\epsilon(t,x) - 1 \to \rho(t,x) - 1 \text{ in } L^\infty\left([0,T], L^2(\Omega)\right) \text{ as } \epsilon\to 0, \tag{3.34}$$

and that

$$\begin{aligned} J^\epsilon(t,x) - (\rho u)(t,x) &= \epsilon\mathrm{Im}\left(\overline{\psi^\epsilon}\,\nabla\psi^\epsilon\right)(t,x) - (\rho u)(t,x) \\ &= \epsilon\mathrm{Im}\left(\overline{\psi^\epsilon}(\nabla - iu)\psi^\epsilon\right)(t,x) + \epsilon\mathrm{Im}\left(\left(|\psi^\epsilon|^2 - \rho\right)u\right)(t,x) \\ &\longrightarrow 0, \text{ in } L^\infty\left([0,T], L^1_{\text{loc}}(\Omega)\right) \text{ as } \epsilon\to 0. \end{aligned} \tag{3.35}$$

This completes the proof of the Theorem. $\square$

4 Appendix: The global existence of solution to (1.1)

For simplicity, let us set $\epsilon = 1$ in (1.1). More precisely, let Ω be an exterior domain of $\mathbb{R}^2$, with $\partial\Omega$ bounded and smooth.Suppose $u^\infty = (u_1^\infty, u_2^\infty)$ is a constant two vector, we consider the global existence of smooth solutions to the following initial boundary value problem:

$$\begin{cases} i\,\partial_t\,\psi = -\dfrac{1}{2}\,\Delta\psi + \left(|\psi|^2 - 1\right)\psi, \qquad x\in\Omega, \quad t\ge 0, \\ \psi(t=0,x) = \psi_0(x), \quad \psi_0(x)\to e^{iu^\infty\cdot x} \quad \text{as} \quad |x|\to\infty, \\ \dfrac{\partial\psi}{\partial\nu}\,|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

Comparing with the problems in [BG] and [TS], one of the main difficulties here is that: since $\psi_0(x)\to e^{iu^\infty\cdot x}$ as $|x|\to\infty$, $\psi_0(\cdot)$ and $\nabla\psi_0(\cdot)\notin L^2(\Omega)$. We will actually prove the existence of more regular solutions than those obtained in [BG] and [TS]. The main result can be stated as the following:

Theorem 4.1. *Let $s \geq 2$ be a positive integer, $\psi_0(x) - e^{iu^\infty \cdot x} \in H^s(\Omega)$. Then* (4.1) *has a unique global smooth solution $\psi(t,x)$ such that $\partial_t^j \, \partial_x^\alpha \left(\psi(t,x) - e^{i(u^\infty \cdot x - \frac{|u^\infty|^2}{2} t)}\right) \in L^\infty\left([0,T],\, H^{s-2j-|\alpha|}(\Omega)\right)$ for all $T < \infty$ and $1 \leq 2j + |\alpha| \leq s$.*

Proof. We use the Galerkin's approximation method to construct the global approximate solutions. As in [H80], we first modify the arguments from page 131 to 133 on [L69] to construct solutions in a bounded domain, then the whole domain Ω. In the arguments for proving the convergence of the approximate solutions, it is essential to obtain a priori estimates of the approximate solutions. For simplicity, we will establish a priori estimates for the smooth solutions of (4.1) instead of the approximate solutions. Therefore in the rest of this section, let us assume that $\psi(t,x)$ is a global smooth solution to (4.1). We divide the proof into two steps.

Step 1. $\boldsymbol{H^2}$ estimate. First of all, as $\psi_0(x) \to e^{iu^\infty \cdot x}$ when $|x| \to \infty$, by the special structure of the Gross-Pitaevskii equation, we expect that for any fixed time $t > 0$, $\psi(t,x)$ approaches $e^{i\left(u^\infty \cdot x - \frac{|u^\infty|^2}{2} t\right)}$ as $|x| \to \infty$. Let us take $\chi(x) \in C^\infty(\mathbb{R}^2)$ be a cut-off function with $\chi(x) = \begin{cases} 0, & \text{for } |x| \leq R \\ 1, & \text{for } \quad |x| \geq 2R, \end{cases}$ where R is big enough such that $\Omega^c \subset B_R(0)$. We denote $B(t,x) = e^{i\left(u^\infty \cdot x - \frac{|u^\infty|^2}{2} t\right)}$ and $A(t,x) = \chi(x)\, B(t,x)$, then by the first equation of (4.1), we have

$$i\, \partial_t(\psi - A) = -\frac{1}{2}\, \Delta(\psi - A) - \left(\frac{1}{2}\, \Delta\chi + iu^\infty \cdot \nabla\chi\right) B + \left(|\psi|^2 - 1\right) \psi, \tag{4.2}$$

Notice that $\frac{\partial(\psi - A)}{\partial\nu}|_{\partial\Omega} = \frac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0$, by taking the complex L^2 inner product of (4.2) with $\partial_t(\psi - A)$ and then take the real part, we obtain

$$\begin{aligned} &\frac{1}{2}\, \frac{d}{dt} \int_\Omega \left(|\nabla(\psi - A)|^2 + (|\psi|^2 - 1)^2\right) dx \\ &\quad = 2\Re \left\{ \frac{d}{dt} \int_\Omega \left(\frac{1}{2}\, \Delta\chi + iu^\infty \nabla\chi\right) B \left(\bar{\psi} - \bar{A}\right) dx \right. \\ &\quad + \frac{|u^\infty|^2}{2}\, i \int_\Omega \left(\frac{1}{2}\, \Delta\chi + iu^\infty \nabla\chi\right) B \left(\bar{\psi} - \bar{A}\right) dx \\ &\quad \left. + \frac{|u^\infty|^2}{2}\, i \int_\Omega \left(|\psi|^2 - 1\right) (\psi - A)\, \bar{B}\, dx \right\}. \end{aligned} \tag{4.3}$$

Note that both $\Delta\chi$ and $\nabla\chi$ have compact support. By integrating the above equation over $[0,t]$, we get

$$\begin{aligned}
&\frac{1}{4}\int_\Omega \left(|\nabla(\psi-A)|^2(+|\psi|^2-1)^2\right)dx \\
&\leq \left\|\left(\psi_0-\chi\, e^{iu^\infty\cdot x}\right)\right\|_{L^2}^2+\left\|\left(|\psi_0|^2-1\right)\right\|_{L^2}^2 \\
&+C_R+\frac{1}{2}\int_\Omega|\psi-A|^2\,dx+\int_0^t\int_\Omega|\psi-A|^2\,dx\,ds.
\end{aligned} \tag{4.4}$$

Next we perform the complex L^2 inner product of (4.2) with $\psi-A$, and then take the imaginary part, we find

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega|\psi-A|^2\,dx \\
&=2\mathcal{I}m\left\{-\int_\Omega\left(\frac{1}{2}\,\Delta\chi+iu^\infty\nabla\chi\right)B\left(\bar\psi-\bar A\right)dx\right. \\
&\left.-\int_\Omega(|\psi|^2-1)(\psi-A)\,\bar B\,dx\right\} \\
&\leq C_R+\int_\Omega|\psi-A|^2\,dx+\int_\Omega(|\psi|^2-1)^2\,dx.
\end{aligned} \tag{4.5}$$

(4.4) and (4.5) together with the Gronwall inequality yield

$$\|(\psi-A)(t,\cdot)\|_{H^1}^2+\left\|(|\psi|^2-1)\right\|_{L^2}^2\leq\left(C_R+\left\|\psi_0-\chi e^{iu^\infty\cdot x}\right\|_{H^1}^2+\left\|(|\psi_0|^2-1)\right\|_{L^2}^2\right)e^t. \tag{4.6}$$

Again since $\frac{\partial(\psi-A)}{\partial\nu}\mid_{\partial\Omega}=\frac{\partial\psi}{\partial\nu}\mid_{\partial\Omega}=0$, one has, by the standard elliptic and Sobolev estimates and (4.2), 4.6), that

$$\begin{aligned}
\|(\psi-A)(t,\cdot)\|_{H^2}&\leq C\left(\|\Delta(\psi-A)\|_{L^2}+\|(\psi-A)\|_{L^2}\right) \\
&\leq C\left(\|\partial_t(\psi-A)\|_{L^2}+\left\|\frac{1}{2}\,\Delta\chi+iu^\infty\nabla\chi\right\|_{L^2}+\left\|(|\psi|^2-1)\psi\right\|_{L^2}+e^t\right) \\
&\leq C\left(\|\partial_t(\psi-A)\|_{L^2}+\|\psi\|_{L^\infty}+e^t\right)\leq C\left(\|\partial_t(\psi-A)\|_{L^2}+\|(\psi-A)\|_{L^\infty}+e^t\right) \\
&\leq C\left(\|\partial_t(\psi-A)\|_{L^2}+e^t\right)+\frac{1}{2}\,\|(\psi-A)(t,\cdot)\|_{H^2}\,.
\end{aligned}$$

The latter implies that

$$\|(\psi-A)(t,\cdot)\|_{H^2}\leq C\left(\|\partial_t(\psi-A)(t,\cdot)\|_{L^2}+e^t\right). \tag{4.7}$$

On the other hand, by applying ∂_t to the first equation of (4.2), and taking the complex L^2 inner product of the resulting equation with $\partial_t(\psi-A)$. (Note that the boundary condition for ψ implies that: $\frac{\partial(\partial_t(\psi-A))}{\partial\nu}\mid_{\partial\Omega}=0$), one has

$$\begin{aligned}
\frac{d}{dt}\int_\Omega|\partial_t(\psi-A)|^2\,dx&=|u^\infty|^2\,\Re\int_\Omega\left(\frac{1}{2}\,\Delta\chi+iu^\infty\nabla\chi\right)B\,\partial_t\left(\bar\psi-\bar A\right)dx \\
&+2\mathcal{I}m\int_\Omega\left((|\psi|^2-1)\,\psi_t+\psi_t|\psi|^2+\psi^2\bar\psi_t\right)\partial_t\left(\bar\psi-\bar A\right)dx.
\end{aligned} \tag{4.8}$$

We observe that

$$\mathcal{I}m \int_\Omega \left(|\psi|^2 - 1\right) \psi_t \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx = -\frac{|u^\infty|^2}{2} \, \Re \int_\Omega \left(|\psi|^2 - 1\right) \chi \, B \, \partial_t \, \left(\bar{\psi} - \bar{A}\right) dx, \tag{4.9}$$

$$\begin{aligned}\mathcal{I}m \int_\Omega \psi_t \, |\psi|^2 \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx &= -\frac{|u^\infty|^2}{2} \, \Re \int_\Omega |\psi|^2 \, \chi \, B \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx \\ &= -\frac{|u^\infty|^2}{2} \, \Re \int_\Omega \left(\left(|\psi|^2 - 1\right) + 1\right) \chi \, B \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx,\end{aligned} \tag{4.10}$$

and that

$$\begin{aligned}\mathcal{I}m \int_\Omega \psi^2 \, \partial_t \, \bar{\psi} \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx = \mathcal{I}m \Bigg\{ &\int_\Omega \psi^2 \left(\partial_t \left(\bar{\psi} - \bar{A}\right)\right)^2 \, dx \\ &+ \frac{i|u^\infty|^2}{2} \int_\Omega \psi^2 \, \chi \, \bar{B} \left(\partial_t \left(\bar{\psi} - \bar{A}\right)\right) dx \Bigg\}.\end{aligned} \tag{4.11}$$

Note

$$\begin{aligned}\int_\Omega \psi^2 \chi \bar{B} \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx &= \int_\Omega \psi(\psi - A) \, \chi \, \bar{B} \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx \\ &+ \int_\Omega (\psi - A) \, \chi^2 \, \partial_t \left(\bar{\psi} - \bar{A}\right) dx + \int_\Omega \chi^3 \, B \partial_t \left(\bar{\psi} - \bar{A}\right) dx.\end{aligned}$$

By summing up (4.9) through (4.11), and using the Gronwall's inequality, we obtain

$$\begin{aligned}\frac{d}{dt} \int_\Omega |\partial_t(\psi - A)|^2 \, dx \le C_R \left(1 + \int_\Omega (|\psi|^2 - 1)^2 \, dx + \int_\Omega |\psi - A|^2 \, dx + \int_\Omega (\chi^3 - \chi)^2 \, dx\right) \\ + 2 \left(\|(\psi - A)\|_{L^\infty}^2 + 1\right) \int_\Omega |\partial_t(\psi - A)|^2 \, dx.\end{aligned} \tag{4.12}$$

Note that $\chi^3 - \chi$ has compact support, then from (4.6), (4.7) and Lemma 2 of [BG], we get

$$\begin{aligned}&\frac{d}{dt} \left(\int_\Omega |\partial_t(\psi - A)|^2 \, dx + 1\right) \\ &\le C \left(e^t + \log\left(1 + \|(\psi - A)(t, \cdot)\|_{H^2}\right)\right) \left(\int_\Omega |\partial_t(\psi - A)|^2 \, dx + 1\right) \\ &\le C \left(e^t + \log\left(e^t + \|\partial_t(\psi - A)\|_{L^2}\right)\right) \left(\int_\Omega |\partial_t(\psi - A)|^2 \, dx + 1\right).\end{aligned} \tag{4.13}$$

The latter together with (4.7) and the Gronwall's inequality shows that

$$\left\|\partial_t^j \, \partial_x^\alpha (\psi - A)(t, \cdot)\right\|_{L^2} \le C \left(t, \|(\psi_0 - e^{iu^\infty \cdot x})\|_{H^2}\right), \quad \forall \quad 1 \le \forall \, 2j + |\alpha| \le 2. \tag{4.14}$$

Step 2. High order estimates. In this step, we are going to prove inductively that

$$\left\|\partial_t^j\,\partial_x^\alpha(\psi-A)(t,\cdot)\right\|_{L^2}\leq C\left(t,\|(\psi_0-e^{iu^\infty\cdot x})\|_{H^s}\right),\quad\forall\quad 1\leq\forall\, 2j+|\alpha|\leq s. \tag{4.15}$$

Let us assume first that (4.15) holds for all j and α with $1\leq 2j+|\alpha|\leq s-1$. We are going to prove that (4.15) also holds for all j and α with $1\leq 2j+|\alpha|\leq s$. If s is an even number, let us denote $j_0=\frac{s}{2},\ j_2=\frac{s-2}{2},\cdots$, if s is an odd number, let us denote $j_1=\frac{s-1}{2},\ j_3=\frac{s-3}{2},\cdots$.

We first apply ∂_t^j to the first equation of (4.2) to get

$$\begin{aligned} i\,\partial_t^{j+1}(\psi-A)=-\frac{1}{2}\,\Delta\,\partial_t^j\,(\psi-A)-\left(-\frac{|u^\infty|^2}{2}i\right)^j\Bigg(\frac{1}{2}\,\Delta\chi&\\ +i\,u^\infty\cdot\nabla\chi\Bigg)B+\partial_t^j\left(\left(|\psi|^2-1\right)\psi\right).& \end{aligned}\tag{4.16}$$

Step 2.1. If s is an even number, let us set $j=j_0$ in (4.16), and take the complex L^2 inner product of the resulting equation with $\partial_t^{j_0}(\psi-A)$. Again by the boundary condition that $\frac{\partial(\partial_t^{j_0}(\psi-A))}{\partial\nu}\mid_{\partial\Omega}=0$, and integration by parts, we find

$$\begin{aligned} \frac{d}{dt}\int_\Omega\left|\partial_t^{j_0}(\psi-A)\right|^2\,dx=2\mathcal{I}m\Bigg\{\int_\Omega\partial_t^{j_0}\left(\left(|\psi|^2-1\right)\psi\right)\partial_t^{j_0}\left(\bar\psi-\bar A\right)dx&\\ -\left(-\frac{|u^\infty|^2}{}2i\right)^j\int_\Omega\left(\frac{1}{2}\,\Delta\chi+iu^\infty\cdot\nabla\chi\right)B\,\partial_t^{j_0}\left(\bar\psi-\bar A\right)dx\Bigg\}.& \end{aligned}\tag{4.17}$$

We can decompose $(|\psi|^2-1)$ as

$$|\psi|^2-1=(\psi-A)\,\bar\psi+A\left(\bar\psi-\bar A\right)+(\chi^2-1), \tag{4.18}$$

therefore by the Leibnitz formula, we get

$$\left|\int_\Omega\partial_t^{j_0}\left((\psi-A)|\psi|^2\right)\partial_t^{j_0}\left(\bar\psi-\bar A\right)dx\right|\leq\sum_{m_1+m_2+m_3=j_0}\left|\int_\Omega\partial_t^{m_1}(\psi-A)\,\partial_t^{m_2}\,\bar\psi\partial_t^{m_3}\,\psi\,\partial_t^{j_0}\left(\bar\psi-\bar A\right)dx\right|. \tag{4.19}$$

If $m_3=j_0$ (resp. $m_2=j_0$), then $m_1=m_2=0$ (resp. $m_1=m_3=0$). Thus one has

$$\begin{aligned} &\int_\Omega(\psi-A)\,\bar\psi\,\partial_t^{j_0}\,\psi\,\partial_t^{j_0}\left(\bar\psi-\bar A\right)dx\\ &\leq\left\|(\psi-A)\,\bar\psi\right\|_{L^\infty}\int_\Omega\left|\partial_t^{j_0}\,(\psi-A)\right|^2\,dx+\left\|\bar\psi\,\partial_t^{j_0}\,A\right\|_{L^\infty}\int_\Omega|\psi-A|\left|\partial_t^{j_0}\left(\bar\psi-\bar A\right)\right|dx. \end{aligned}\tag{4.20}$$

The other terms in (4.19) can be estimated by

$$
\begin{aligned}
&\le C \sum_{l \le j_0 - 1} \left\|\partial_t^l \psi\right\|_{L^\infty}^2 \left(\sum_{m \le j_0} \int_\Omega \left|\partial_t^m (\psi - A)\right|^2 \, dx\right) \\
&\qquad\qquad \le C \sum_{l \le j_0 - 1} \left(1 + \left\|\partial_t^l (\psi - A)\right\|_{L^\infty}^2\right) \left(\sum_{m \le j_0} \int_\Omega \left|\partial_t^m (\psi - A)\right|^2 \, dx\right).
\end{aligned} \tag{4.21}
$$

By the inductive hypothesis, we have $\left\|\partial_t^{j_0-1} (\psi - A)(t, \cdot)\right\|_{H^1}$, $\left\|\partial_t^l (\psi - A)(t, \cdot)\right\|_{H^3} \le C\left(t, \|(\psi_0 - e^{iu^\infty \cdot x}\|_{H^{s-1}}\right)$, for all $l \le j_0 - 2$, which together with (4.17–4.21) implies that

$$
\frac{d}{dt} \int_\Omega \left|\partial_t^{j_0} (\psi - A)\right|^2 \, dx \le C(t) \left(1 + \|\partial_t^{j_0-1} (\psi - A)\|_{L^\infty}\right) \left(1 + \int_\Omega |\partial_t^{j_0} (\psi - A)|^2 \, dx\right). \tag{4.22}
$$

While by the standard elliptic theory, and (4.16), we have

$$
\begin{aligned}
&\left\|\partial_t^{j_0-1} (\psi - A)(t, \cdot)\right\|_{H^2} \le C \left(\|\Delta \, \partial_t^{j_0-1} (\psi - A)\,\|_{L^2} + \left\|\partial_t^{j_0-1} (\psi - A)\right\|_{L^2}\right) \\
&\le C \left(\left\|\partial_t^{j_0} (\psi - A)\right\|_{L^2} + \left\|\left(\frac{1}{2} \, \Delta \, \chi + iu^\infty \nabla \chi\right)\right\|_{L^2} + \left\|\partial_t^{j_0-1} \left(\left(|\psi|^2 - 1\right) \psi\right)\right\|_{L^2} + C(t)\right) \\
&\le C \left(\left\|\partial_t^{j_0} (\psi - A)\right\|_{L^2} + C(t)\right).
\end{aligned} \tag{4.23}
$$

Combining (4.22–4.23) and Lemma 2 in [BG], we find

$$
\frac{d}{dt} \int_\Omega \left|\partial_t^{j_0} (\psi - A)\right|^2 \, dx \le C(t) \left(1 + \log\left(C(t) + \left\|\partial_t^{j_0} (\psi - A)\right\|_{L^2}\right)\right) \left(1 + \left\|\partial_t^{j_0} (\psi - A)\right\|_{L^2}\right). \tag{4.24}
$$

From the Gronwall inequality, one obtains

$$
\left\|\partial_t^{j_0} (\psi - A)(t, \cdot)\right\|_{L^2} \le C\left(t, \left\|\left(\psi_0 - e^{iu^\infty \cdot x}\right)\right\|_{H^s}\right). \tag{4.25}
$$

Consequently, by (4.23), we get

$$
\left\|\partial_t^{j_0-1} \, \partial_x^\alpha (\psi - A)(t, \cdot)\right\|_{L^2} \le C\left(t, \left\|\left(\psi_0 - e^{iu^\infty \cdot x}\right)\right\|_{H^s}\right), \quad |\alpha| \le 2. \tag{4.26}
$$

On the other hand, from (4.16), we have

$$
\begin{cases}
\frac{1}{2} \, \Delta \, \partial_t^{j_0-2} (\psi - A) = -i \, \partial_t^{j_0-1} (\psi - A) + \left(-\frac{|u^\infty|^2}{2} i\right)^{j_0-2} \left(\frac{1}{2} \, \Delta \, \chi\right. \\
\left. \quad + \, iu^\infty \nabla \chi\right) B + \partial_t^{j_0-2} \left(\left(|\psi|^2 - 1\right) \psi\right) \\
\frac{\partial \left(\partial_t^{j_0-2} (\psi - A)\right)}{\partial \nu}\Big|_{\partial\Omega} = 0.
\end{cases}
$$

Then it follows from (4.26) and (4.19–4.21), that

$$\left\|\Delta\, \partial_t^{j_0-2}\,(\psi - A)\right\|_{H^2} \le C\left(t, \left\|\left(\psi_0 - e^{iu^\infty\cdot x}\right)\right\|_{H^s}\right).$$

The last estimate together with the standard elliptic theory implies

$$\left\|\partial_t^{j_0-2}\partial_x^\alpha(\psi - A)\right\|_{L^2} \le C\left(t, \left\|\left(\psi_0 - e^{iu^\infty\cdot x}\right)\right\|_{H^s}\right), \quad |\alpha| \le 4. \tag{4.27}$$

Following the similar derivation as that for (4.25)–(4.27), we prove (4.15) when s is an even number.

Step 2.2. Suppose s is an odd number and let us set $j = j_1$ in (4.16). We take the complex L^2 inner product of the resulting equation with $\partial_t^{j_1}(\psi - A)$ and use integration by parts, after taking the real part we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\partial_t^{j_1}(\psi - A)|^2\,dx + 2\Re\int_\Omega \partial_t^{j_1}\left(\left(|\psi|^2-1\right)\psi\right)\partial_t^{j_1+1}\left(\bar\psi - \bar A\right)dx \\ - 2\Re\left(\left(-\frac{|u^\infty|^2}{2}i\right)^{j_1}\int_\Omega\left(\frac{1}{2}\,\Delta\,\chi + iu^\infty\nabla\chi\right)B\,\partial_t^{j_1+1}\left(\bar\psi - \bar A\right)dx\right) = 0.\end{aligned} \tag{4.28}$$

By using (4.18) and the Lebnitz formula, we get

$$\begin{aligned}\partial_t^{j_1}\left(\left(|\psi|^2-1\right)\psi\right) &= \left(|\psi|^2-1\right)\partial_t^{j_1}\psi + \partial_t^{j_1}(\psi - A)|\psi|^2 + A\partial_t^{j_1}\left(\bar\psi - \bar A\right)\psi \\ &+ \sum_{\substack{m_1+m_2+m_3=j_1\\ \max(m_k)\le j_1-1}}\left(\partial_t^{m_1}(\psi - A)\,\partial_t^{m_2}\bar\psi\,\partial_t^{m_3}\psi + \partial_t^{m_1}A\,\partial_t^{m_2}\left(\bar\psi - \bar A\right)\partial_t^{m_3}\psi\right).\end{aligned}$$

By using integration by parts on the t variable in (4.28), we find

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\int_\Omega\left|\nabla\,\partial_t^{j_1}(\psi - A)\right|^2\,dx + \frac{d}{dt}\int_\Omega\left(\left(2|\psi|^2-1\right)\left|\partial_t^{j_1}(\psi - A)\right|^2 + \Re\left(A\psi\left(\partial_t^{j_1}\left(\bar\psi - \bar A\right)^2\right)\right)\right)dx \\ &+ \sum_{\substack{m_1+m_2+m_3=j_1\\ \max(m_k)\le j_1-1}}\frac{d}{dt}\int_\Omega\left(\partial_t^{m_1}(\psi - A)\,\partial_t^{m_2}\,\bar\psi\,\partial_t^{m_3}\,\psi + \partial_t^{m_1}\,A\,\partial_t^{m_2}\left(\bar\psi - \bar A\right)\partial_t^{m_3}\,\psi\right)\partial_t^{j_1}\left(\bar\psi - \bar A\right)dx \\ &+ 2\frac{d}{dt}\,\Re\left\{\left(-\frac{|u^\infty|^2}{}2i\right)^{j_1}\int_\Omega\left(\frac{1}{2}\,\Delta\chi + iu^\infty\nabla\chi\right)B\left(\partial_t^{j_1}\left(\bar\psi - \bar A\right)\right)dx\right\} \\ &\le C_R\left\|\partial_t^{j_1}(\psi - A)\right\|_{L^2} + \sum_{\substack{m_1+m_2\le j_1+1\\ \max(m_1,m_2)\le j_1-1}}\left(\left\|\partial_t^{m_1}\,\bar\psi\,\partial_t^{m_2}\,\bar\psi\right\|_{L^\infty} + \left\|\partial_t^{m_1}\,A\,\partial_t^{m_2}\,\psi\right\|_{L^\infty}\right)\cdot \\ &\left(\sum_{m\le j_1}\left\|\partial_t^m\,(\psi - A)\right\|_{L^2}^2\right) \\ &\le C(t)\left(1 + \sum_{\substack{m_1+m_2\le j_1+1\\ \max(m_1,m_2)\le j_1-1}}\left(\left\|\partial_t^{m_1}\,\psi\right\|_{L^\infty}\left\|\partial_t^{m_2}\,\bar\psi\right\|_{L^\infty} + \left\|\partial_t^{m_1}\,\psi\right\|_{L^\infty}\right)\right),\end{aligned} \tag{4.29}$$

where we used the facts that $2j_1 = s-1$ and the inductive assumption. We also note that $2m_k + |\alpha| \leq 2(j_1-1) + |\alpha| \leq s-1$, hence $|\alpha| \leq 2$. By the inductive assumption, we therefore have

$$\|\partial_t^{m_k}(\psi - A)(t,\cdot)\|_{H^2} \leq C\left(t, \left\|\left(\psi_0 - e^{iu^\infty\cdot x}\right)\right\|_{H^{s-1}}\right), \quad k = 1, 2.$$

Finally, we integrate (4.29) over $[0,t]$ and use the inductive assumption again, to obtain

$$\left\|\nabla \partial_t^{j_1}(\psi - A)(t,\cdot)\right\|_{L^2} \leq C\left(t, \|(\psi_0 - e^{iu^\infty\cdot x})\|_{H^s}\right). \tag{4.30}$$

With (4.30) and the equation

$$\begin{cases} \dfrac{1}{2}\,\Delta\,\partial_t^{j_1-1}(\psi - A) = -i\,\partial_t^{j_1}(\psi - A) + \left(-\dfrac{|u^\infty|^2}{2}i\right)^{j_1-1}\left(\dfrac{1}{2}\triangle\chi\right. \\ \left.+iu^\infty\nabla\chi\right)B + \left(\partial_t^{j_1-1}(|\psi|^2-1)\psi\right), \\ \dfrac{\partial(\partial_t^{j_1}(\psi - A)}{\partial\nu}\,|_{\partial\Omega} = 0, \end{cases}$$

we follow the similar arguments as in deriving (4.25)–(4.27), to conclude

$$\left\|\partial_t^{j_1-1}\,\partial_x^\alpha(\psi - A)(t,\cdot)\right\|_{L^2} \leq C\left(t, \|(\psi_0 - e^{iu^\infty x})\|_{H^s}\right), \quad \forall\ |\alpha| \leq 3. \tag{4.31}$$

Similar to the proof of (4.31), we can prove (4.15) when s is an odd number. This completes the proof of the Theorem. □

Acknowledgments.

Part of the work was done while both authors were visiting the Mathematics Center at the Zhejiang University, the authors would like to thank the hospitality of the Center. F. Lin is partially supported by the NSF Grant, DMS 0201443 and the Chang-Jiang Professorship fund at the Zhejiang University. Ping Zhang is partially supported by the NSF of China under Grant 10131050 and 10276036, and the innovation grant from the Chinese Academy of Sciences.

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