LANGLANDS FUNCTORIALITY AND SOME APPLICATIONS

Dihua Jiang
University of Minnesota

I shall talk about the recent progress on Langlands Functoriality Conjecture in the theory of automorphic forms, in particular these directly related to my recent work.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Automorphic Forms and L-functions

1.1. The Riemann zeta function. $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $Re(s) > 1$.

'Nice' Properties:

(MC) It has meromorphic continuation to $\mathbb{C}$ and has a simple pole at $s = 1$ with residue one.

(FE) Let $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Functional Equation: $\Lambda(s) = \Lambda(1-s)$.

(EP) It has the Euler product decomposition $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$.

(RH) The nontrivial zeros of $\zeta(s)$ are conjecturally on the critical line $Re(s) = \frac{1}{2}$.

(SV) For $k > 0$, $\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}B_{2k}}{(2k)!} \cdot \pi^{2k}$, but almost nothing is known about $\zeta(2k+1)$.

1.2 L-functions:

Let $\mathcal{M} = \{a_n \in \mathbb{C} \mid n \in \mathbb{N}\}$. We may form

$$L(\mathcal{M}, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Assume it absolutely converges for $\Re(s) > s_0$.

Question: When does $L(\mathcal{M}, s)$ have nice properties as (MC), (FE), and/or (EP)?

General Expectation: If $\mathcal{M}$ comes from arithmetic geometry or number theory, then $L(\mathcal{M}, s)$ should have the nice properties.

Langlands Philosophy: If $\mathcal{M}$ is attached to automorphic forms f, then $L(\mathcal{M}, s) = L(f, s)$ should have the nice properties.

1.3. Examples:

(1) **Taniyama-Shimura-Weil Conjecture:** Let $\mathcal{E}$ be an elliptic curve defined over $\mathbb{Q}$. There exists an new form $f_{\mathcal{E}}$ of weight two, such that

$$L(\mathcal{E}, s) = L(f_{\mathcal{E}}, s).$$

It follows that $L(\mathcal{E}, s)$ is a Mellin transform of f_E, and hence $L(\mathcal{E}, s)$ has the nice properties.

It is now a Theorem of A. Wiles (1994), of Taylor-Wiles (1994), of F. Diamond (1996), of Conrad-Diamond-Taylor (1999), and of Breuil-Conrad-Diamond-Taylor (2001).

(2) $\zeta(s) = L(\chi_0, s)$ where χ_0 is the trivial automorphic form of $GL(1)$.

(3) **Artin Conjecture:**

Let ρ: $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(V)$ be a finite dimensional complex representation. Then $L(\rho, s)$ has the nice properties. Moreover, $L(\rho, s)$ is entire if ρ is non-trivial and irreducible.

Langlands Reciprocity Conjecture: For the given ρ, there exists an automorphic form f_ρ of $GL(n)$ ($n = \dim V$), such that $L(\rho, s) = L(f_\rho, s)$.

The theory of Godement and Jacquet shows that $L(f_\rho, s)$ has the nice properties.

See R. Taylor's ICM02 report of detailed discussion on this problem.

Of course L. Lafforgue proved this global Langlands conjecture for $GL(n)$ over function fields.

1.4. Euler Product Formula.

Recall (EP): $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. A famous consequence of the formula is

Theorem (Euler). $\zeta(s)$ *has a pole at* $s = 1$ *iff there exist infinitely many prime numbers in* $\mathbb{Z}$.

It seems no longer elementary that

$$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\prod_p \frac{1}{1-p^{-s}}$$

is related to harmonic analysis over local field and to the local-global principle in number theory.

1.5. Local Fields: $\Omega := \{\infty, 2, 3, 5, \cdots, p, \cdots\}$, the set of all places of $\mathbb{Q}$. For any $v \in \Omega$, the local field at v is $\mathbb{Q}_v = \begin{cases} \mathbb{R} & \text{if } v = \infty, \\ \mathbb{Q}_p & \text{if } v = p. \end{cases}$ Here $\mathbb{Q}_p$ is the field of p-adic numbers, consisting of $x = \sum_{m=n}^{\infty} a_m p^m$ $(0 \leq a_m < p)$. The absolute value of x is: $|x|_p := p^{-n}$.

Then $\mathbb{Q}_v$'s are locally compact and $\mathbb{Q}$ is dense in $\mathbb{Q}_v$. Note that $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \ : \ |x|_p \leq 1\}$ is an open compact topological subring in $\mathbb{Q}_p$.

Theorem. *Any locally compact topological field, in which $\mathbb{Q}$ is dense, is isomorphic to $\mathbb{Q}_v$ for some $v \in \Omega$.*

1.6. Adeles: are defined as follows:

$$\mathbb{A} := \{(x_v) \in \prod_v \mathbb{Q}_v \ : \ |x_p|_p \leq 1, \text{ for almost all } p\}$$

which can also be expressed as

$$\mathbb{A} := \varinjlim_S \mathbb{A}(S)$$

where, for a finite subset $S \subset \Omega$, including $v = \infty$, one defines $\mathbb{A}(S) = \prod_{v \in S} \mathbb{Q}_v \times \prod_{p \notin S} \mathbb{Z}_p$, which has the product topology, so that $\mathbb{A}$ has the direct limit topology.

Theorem. *$\mathbb{Q}$ is a discrete subgroup in $\mathbb{A}$ and the quotient $\mathbb{A}/\mathbb{Q}$ is compact.*

Theorem (Tate).

(1) *For* $v \in \Omega$, $\exists$ *a Schwartz function* ϕ_v, *s.t.*

$$\int_{\mathbb{Q}_v^\times} \phi_v(x)|x|_v^s d^\times x = \begin{cases} \frac{1}{1-p^{-s}} & \text{if } v = p, \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v = \infty. \end{cases}$$

(2) $\exists$ *a Schwartz function* $\phi = \otimes_v \phi_v$ *on* $\mathbb{A}$, *s.t.*

$$\int_{\mathbb{A}^\times} \phi(x)|x|_{\mathbb{A}}^s d^\times x = \Lambda(s).$$

(3) *(FE) for* $\Lambda(s)$ *follows from the Poisson Summation Formula.*

1.7. Algebraic Groups. Algebraic groups G are algebraic varieties with group operations which are morphisms of algebraic varieties.

G is reductive if representations of G are completely reducible.

Examples: $G = GL_n,\ SL_n,\ SO_{2n+1},\ Sp_{2n}$, SO_{2n} or Exceptional groups. If one takes

$$J_{2n+1} := \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix},$$

$SO_{2n+1} = \{g \in GL_{2n+1} \mid {}^t g J g = J, \det g = 1\}$.

Assume that reductive groups G split over $\mathbb{Q}$.

For each $v \in \Omega$, $K_v = G(\mathbb{Z}_v)$ is a maximal compact subgroup if $v < \infty$; and K_∞ is the maximal compact subgroup of $G(\mathbb{R})$ associated to a Cartan involution. Then $K = \prod_v K_v$ is a maximal compact subgroup of $G(\mathbb{A})$.

Theorem. *(1) $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$ via diagonal embedding, and the quotient*

$$Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$$

has finite volume, where Z_G is the center of G.

(2) (Iwasawa Decomposition) $G(\mathbb{A}) = P(\mathbb{A})K$, where P is a $\mathbb{Q}$-parabolic subgroup of G

1.8. Automorphic Forms.

Definition. *An automorphic form is a* $\mathbb{C}$*-valued smooth function on* $G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$ *satisfying following conditions:*

(1) $f(\gamma g) = f(g)$ *for* $\gamma \in G(\mathbb{Q})$ *and* $g \in G(\mathbb{A})$*,*

(2) f *is* K*-finite, with the right translation,*

(3) f *is* $\mathfrak{z}$*-finite, where* $\mathfrak{z}$ *is the center of the universal enveloping algebra* $\mathcal{U}(\mathfrak{g}_\infty)$*,*

(4) For $g_f \in G(\mathbb{A}_f)$*,* $g_\infty \mapsto f(g_\infty g_f)$ *is a slowly increasing function on* G_∞ $(|f(g_\infty g_f)| \leq c||g_\infty||^r)$*.*

$\mathcal{A}(G)$ denotes the set of all automorphic forms on $G(\mathbb{A})$. It has a structure of $(\mathfrak{g}_\infty, K) \times G(\mathbb{A}_f)$-module.

Definition. *An irreducible unitary representation* (V, π) *of* $G(\mathbb{A})$ *is* automorphic *if the space of smooth vectors,* (V^∞, π^∞) *of* (V, π) *is isomorphic to an irreducible subquotient of* $\mathcal{A}(G)$ *as* $(\mathfrak{g}_\infty, K) \times G(\mathbb{A}_f)$*-modules.*

Examples: $L^2(G) = L^2(Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$. Any irreducible subrepresentation (V, π) in $L^2(G)$ is automorphic.

Holomorphic cusp forms on the upper half plane generate irreducible subrepresentations in $L^2(GL(2))$. Hence they generate irreducible cuspidal automorphic representations of $GL(2)$.

Theorem (D. Flath). *Any irreducible unitary representation* π *of* $G(\mathbb{A})$ *can be written as a restricted tensor product* $\pi = \otimes_v \pi_v$ *where* π_v *is an irreducible unitary representation of* $G(\mathbb{Q}_v)$, *and for almost all finite places* v, π_v *has* K_v*-invariant vectors, i.e.* $\pi_v^{K_v} \neq 0$, *where*

$$\pi_v^{K_v} = \{\xi \in \pi_v : \pi_v(K_v)\xi = \xi\}.$$

The reason is that all local groups $G(\mathbb{Q}_v)$ are locally compact topological groups of type I in the sense that their corresponding C^*-algebras are of type I (by a Theorem of Harish-Chandra and a Theorem of J. Bernstein).

1.9. Theory of Spherical Functions.

(1) $\dim \pi_v^{K_v} \leq 1$.

(2) If $\pi_v^{K_v} \neq 0$ (π_v is spherical), there is a unramified character χ_v of $T(\mathbb{Q}_v)$ s.t. π_v is the irreducible spherical constituent of $Ind_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v)$, where $B = TU$ is a Borel subgroup of G.

(3) The K_v-invariant vector of π_v is characterized by a semi-simple conjugacy class t_{π_v} in the Langlands dual group ${}^LG$ (Satake parameter).

Example: $G = GL(m)$. The Satake parameter for $Ind_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_v)$ is

$$t_{\chi_v} = diag(q^{-s_1}, q^{-s_2}, \cdots, q^{-s_m}) \in GL_m(\mathbb{C}),$$

if $\chi_v(diag(t_1, t_2, \cdots, t_m)) = |t_1|^{s_1} \ldots |t_m|^{s_m}$.

1.10. Langlands Dual Groups.

For split reductive algebraic groups, the Langlands dual groups are complex algebraic groups which can be described in terms of combinatorial data of G.

$$\begin{array}{ccc} G & \Longleftrightarrow & (X, \Delta; X^\vee, \Delta^\vee) \\ \updownarrow & & \updownarrow \\ {}^L G & \Longleftrightarrow & (X^\vee, \Delta^\vee; X, \Delta) \end{array}$$

For example, we have

G	${}^L G$
$GL(m)$	$GL(m, \mathbb{C})$
$SL(m)$	$PGL(m, \mathbb{C})$
$SO(2n+1)$	$Sp(2n, \mathbb{C})$
$Sp(2n)$	$SO(2n+1, \mathbb{C})$
$SO(2n)$	$SO(2n, \mathbb{C})$
G_2	$G_2(\mathbb{C})$

1.11. Automorphic L-functions.

Definition. *Let $\pi = \otimes_v \pi_v$ be an irreducible unitary automorphic representation of $G(\mathbb{A})$ and r be a finite dimensional representation of the dual group ${}^L G$. For any unramified local component π_v ($v < \infty$), the local L-factor is defined by*

$$L(\pi_v, r, s) := [\det(I - r(t_{\pi_v}) q_v^{-s})]^{-1}.$$

The Langlands L-function attached to (π, r) is

$$L(\pi, r, s) := \prod_{v \in S} L(\pi_v, r, s) \times \prod_{v \notin S} L(\pi_v, r, s)$$

where S is a finite subset of Ω such that π_v is unramified if $v \notin S$.

For $v \in S$, it is a difficult problem to define $L(\pi_v, r, s)$.

Theorem (Langlands). *For any pair* (π, r) *as given in the Definition, the Euler product* $L(\pi, r, s)$ *absolutely converges for* $\Re(s)$ *large.*

Conjecture (Langlands). *Automorphic L-functions* $L(\pi, r, s)$ *have meromorphic continuation to* $\mathbb{C}$ *(MC) with finitely many poles in* $\mathbb{R}$*, and enjoy the functional equation (FE):*

$$L(\pi, r, s) = \epsilon(\pi, r, s) L(\pi^\vee, r, 1 - s).$$

Remark: This conjecture has been verified in many cases based on the spectral Theory of automorphic forms and Eisenstein series, by both the Rankin-Selberg method and the Langlands-Shahidi method. The significance of the conjecture is well known. For instance, it implies the Ramanujan Conjecture for Maass forms.

2. Langlands Conjectures

2.1. Langlands Functoriality: Let G and H be split reductive groups over k. For any admissible map, $\rho : {}^LG \to {}^LH$, there is a functorial lift ρ^* ('many to many' map) from the set $\Pi(G)$ of equivalence classes of irreducible automorphic representations π of $G(\mathbb{A})$ to $\sigma \in \Pi(H)$ s.t.

$$L(\pi, \rho \circ r, s) = L(\rho^*(\pi), r, s)$$

Example: (1) $\rho : Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ should yield a functorial lift from SO_{2n+1} to GL_{2n}.

(2) If $G = 1$, $H = GL(m)$, it is the Langlands reciprocity conjecture on Galois representations.

2.2. Recent Progress on Functoriality

(1) **Classical Groups to $GL(N)$.** For classical groups $G = SO_{2n+1}, Sp_{2n}$, or SO_{2n}, the dual groups are $Sp_{2n}(\mathbb{C}), SO_{2n+1}(\mathbb{C})$, or $SO_{2n}(\mathbb{C})$, respectively.

Langlands Functoriality Conjecture asserts that automorphic representations of $G(\mathbb{A})$ should have lifting to automorphic representations of $GL_N(\mathbb{A})$, where $N = 2n$ if $G = SO_{2n+1}$ or SO_{2n}, and $N = 2n+1$ if $G = Sp_{2n}$.

A first **General Result** was obtained by J. Cogdell, H. Kim, I. Piatetski-Shapiro, and F. Shahidi (Publ. Math. IHES (2001)): *A weak Langlands functorial lift from irreducible generic cuspidal automorphic representation* π *of* SO_{2n+1} *to* $\sigma = \rho^*(\pi)$ *of* GL_{2n} *exists, i.e. for all archimedean places or all finite places where* π_v *is unramified,* $\pi_v \mapsto \sigma_v$ *is local Langlands functorial* $(L(\pi_v, s) = L(\sigma_v, s))$.

This weak lift was proved to be the global Langlands functorial lift by D. Soudry and D.-H. Jiang (Shalikafest 2002) and by H. Kim (Trans. Amer. Math. Soc. (2002)), using different methods.

D. Ginzburg, S. Rallis, and D. Soudry characterize explicitly the **image** of this functorial lift (Internat. Math. Res. Notices (2001)). D. Soudry and D.-H. Jiang prove this functorial lift is **injective** when restricted to the generic automorphic representations (Ann. of Math. (2003))

The existence result has been extended to SO_{2n} and Sp_{2n} in a recent preprint of Cogdell; Kim; Piatetski-Shapiro; Shahidi. The work of Ginzburg-Rallis-Soudry has been carried over to these cases. The generalization of the work of Jiang-Soudry to these cases is our work in progress.

For non-generic cases no general result is known, except some lower rank cases.

(2) **Some Interesting Lower Rank Cases**

Kim-Shahidi (2000) The symmetric cube and forth power lift for $GL(2)$ exists and the cuspidality of the lifts are studied (The symmetric square lift for $GL(2)$ was due to Gelbart-Jacquet). We refer to Shahidi's ICM02 talk for striking applications in number theory.

The weak functorial lift from G_2 to $GSp(6)$ exists (Ginzburg-Jiang (2001) for split G_2 (generic cases); and Gross-Savin (1998) for anisotropic G_2).

Base Change for cyclic field extensions for $GL(m)$ by Arthur-Clozel (1989), and for certain unitary groups by Labesse-Clozel (1999).

3. Global Applications

Rigidity Theorem (Jiang-Soudry (2003)) Two irreducible generic cuspidal automorphic representations of SO_{2n+1} are isomorphic if they are isomorphic at almost all local places (For $GL(m)$, it was proved by H. Jacquet and J. Shalika in 1970's and is called (for $GL(n)$) *The Strong Multiplicity One Theorem*).

Conjecture of Ramkrishnan-Prasad: If an irreducible cuspidal automorphic representation Σ of $GL(2n)$ has the property that the exterior square L-function $L(\Sigma, \Lambda, s)$ has a pole at $s = 1$, then every local component of Σ is symplectic (Jiang-Soudry (2003)).

Positivity of the Central L-values (Lapid-Rallis, Ann. of Math. (2003)) If an irreducible unitary cuspidal automorphic representation π of $GL_n(\mathbb{A})$ is symplectic, then $L(\pi, \frac{1}{2}) \geq 0$.

More generally, if π is symplectic of $GL(n)$ and π' is orthogonal of $GL(m)$, then $L(\pi \times \pi', \frac{1}{2}) \geq 0$ (Lapid, 2002).

Question: When does $L(\pi \times \pi', \frac{1}{2})$ non-zero?

Ginzburg-Jiang-Rallis (two preprints (2003)) characterizes the non-vanishing of $L(\pi \times \pi', \frac{1}{2})$ in terms of Gross-Prasad periods in general. In their paper (Canad. J. Math. (1992) and (1994)), Gross-Prasad formulated a conjecture on local version of such a characterization.

Final Remarks: The Langlands functorial conjecture asserts the basic relations among automorphic forms on different groups. Once the existence of such relations has been established, one expects many significant applications to come, both in the global theory and in the local theory. For technical reasons, I can not mention all the known applications. For example, the application to the local Langlands conjecture for classical groups has not been discussed here (see Jiang-Soudry (2002), (2003) for details).

In Friday's lecture I will give some more details on my work mentioned above. (THANK YOU!)