Lower Tail Probabilities and Related Problems

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1. Lower Tail Probabilities

Let $\{X_t,\ t \in T\}$ be a real valued Gaussian process indexed by T with $\mathbb{E}\, X_t = 0$.

$$\mathbb{P}\Big(\sup_{t \in T} (X_t - X_{t_0}) \le x \Big) \text{ as } x \to 0$$

where $t_0 \in T$.

Examples:

(a) Csaki, Khoshnevisan and Shi (2000):

Let $W(s,t)$ be the two dimensional Brownian sheet. Then for $x > 0$ small

$$\ln \mathbb{P}\Big(\sup_{0\le s,t\le 1} W(s,t) \le x \Big) \succeq -\ln^2(1/x)$$

$$\ln \mathbb{P}\Big(\sup_{0\le s,t\le 1} W(s,t) \le x \Big) \preceq -\frac{\ln^2(1/x)}{\ln\ln(1/x)}.$$

(b) Capture time of Brownian pursuits (Bramson and Griffeath (1991)):

Let $W_0, W_1, \cdots, W_n$ be independent standard Brownian motions. Define

$$\tau_n = \inf \left\{ t > 0 : \max_{1\le k\le n} W_k(t) = W_0(t) + 1 \right\}.$$

When is $\mathbb{E}\left(\tau_n\right)$ finite?

Note that for any $a > 0$, by Brownian scaling,

$$\begin{aligned} &\mathbb{P}(\tau_n > t) \\ &= \mathbb{P}\Big(\max_{1\le k\le n} \sup_{0\le s\le t} (W_k(s) - W_0(s)) < 1 \Big) \\ &= \mathbb{P}\Big(\max_{1\le k\le n} \sup_{0\le s\le 1} (W_k(s) - W_0(s)) < t^{-1/2} \Big). \end{aligned}$$

Thus the problem is really a lower tail probability problem.

DeBlassie (1987):

$$\mathbb{P}\{\tau_n > t\} \sim ct^{-\gamma_n} \quad \text{as} \quad t \to \infty.$$

Bramson and Griffeath (1991): $\mathbb{E}\,\tau_3 = \infty$

Conjecture: $\mathbb{E}\,\tau_4 < \infty$.

Li and Shao (2001): $\mathbb{E}\,\tau_5 < \infty$.

(c) The probability that a random polynomial has no real root

(Dembo, Poonen, Shao and Zeitouni (2002))

$$\mathbb{P}\Big(\sum_{i=0}^{n} Z_i x^i < 0 \ \forall \ x \in \mathbb{R}^1 \Big) = n^{-b+o(1)}$$

where n is even, Z_i are i.i.d. $N(0,1)$, and

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le 0 \Big)$$

where X_t is a centered stationary Gaussian process with

$$\mathbb{E}\, X_s X_t = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}}$$

A General Result

Let $X = \{X_t,\ t \in T\}$ be a real valued Gaussian random process indexed by T with mean zero. Define the L^2-metric

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s,t \in T.$$

For every $\varepsilon > 0$ and a subset A of T, let $N(A,\varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover A. For $t \in T$ and $h > 0$, let

$$B(t,h) = \{s \in T : d(t,s) \le h\}$$

and define

$$Q = \sup_{h>0} \sup_{t \in T} \int_0^\infty (\ln N(B(t,h), \varepsilon h))^{1/2} d\varepsilon$$

For $\theta = 1000(1+Q)$, define

$$\begin{aligned} \mathcal{A}_{-1} &= \{t \in T : d(t,t_0) \le \theta^{-1} x\}, \\ \mathcal{A}_k &= \{t \in T : \theta^{k-1} x < d(t,t_0) \le \theta^k x\}, \end{aligned}$$

where $0 \le k \le L$, $L = 1 + [\ln_\theta(D/x)]$ and $D = \sup_{t \in T} d(t,t_0)$.

Let

$$\begin{aligned} N_k(x) &= N(\mathcal{A}_k,\ \theta^{k-2} x) \ \text{ for } \ k = 0,1,\cdots,L \\ N(x) &= 1 + \sum_{0 \le k \le L} N_k(x). \end{aligned}$$

Li and Shao (2003):

- *Assume that* $Q < \infty$ *and*

$$\mathbb{E}\left((X_s - X_{t_0})(X_t - X_{t_0})\right) \geq 0 \quad \textit{for} \quad s, t \in T$$

Then

$$\mathbb{P}\Big(\sup_{t \in T} X_t - X_{t_0} \leq x\Big) \geq e^{-N(x)}$$

- *For* $x > 0$, *let* $s_i \in T$, $i = 1, ..., M$ *be a sequence such that for every* i

$$\sum_{j=1}^{M} |\,Corr(X_{s_i} - X_{t_0}, X_{s_j} - X_{t_0})| \leq 5/4$$

and

$$d(s_i, t_0) = (\mathbb{E}\,|X_{s_i} - X_{t_0}|^2)^{1/2} \geq x/2.$$

Then

$$\mathbb{P}\Big(\sup_{t \in T} X_t - X_{t_0} \leq x\Big) \leq e^{-M/10}.$$

Some Special Cases

- *Let* $\{X(t), t \in [0,1]^d\}$ *be a centered Gaussian process with* $X(0) = 0$ *and stationary increments, that is*

  $$\forall\, t, s \in [0,1]^d, \quad \mathbb{E}\,(X_t - X_s)^2 = \sigma^2(||t-s||).$$

  If there are $0 < \alpha \leq \beta < 1$ *such that*

  $$\sigma(h)/h^{\alpha} \uparrow, \quad \sigma(h)/h^{\beta} \downarrow \qquad (*)$$

  Then there exist $0 < c_1 \leq c_2 < \infty$ *depending only on* α*,* β *and* d *such that for* $0 < x < 1/2$

  $$-c_2 \ln \frac{1}{x} \leq \ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} X(t) \leq \sigma(x) \Big) \leq -c_1 \ln \frac{1}{x}.$$

  In particular, for the fractional Levy's Brownian motion $L_{\alpha}(t)$ of order α, i.e. $L_{\alpha}(0) = 0$ and

  $$\mathbb{E}\,(L_{\alpha}(t) - L_{\alpha}(s))^2 = ||t-s||^{\alpha},$$

  $$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} L_{\alpha}(t) \leq x \Big) \approx -\ln \frac{1}{x}.$$

- Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and

  $$\mathbb{E}\,(X_t X_s) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)).$$

If there are $0 < \alpha \leq \beta < 1$ such that

$$\sigma(h)/h^\alpha \uparrow, \quad \sigma(h)/h^\beta \downarrow$$

Then

$$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} X(t) \leq \sigma^d(x) \Big) \approx -\ln^d \frac{1}{x}.$$

In particular, for d-dimensional Brownian sheet $W(t)$

$$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} W(t) \leq x \Big) \approx -\ln^d \frac{1}{x}$$

and more generally

$$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} B_\alpha(t) \leq x \Big) \approx -\ln^d \frac{1}{x}$$

- Open question:
  Can the assumption $(*)$ be replaced by

$$c_1 \sigma(h) \leq \sigma(2h) \leq c_2 \sigma(h)$$

  for some $c_2 \geq c_1 > 1$?

2. Lower Tail Probabilities for Stationary Gaussian Processes

Let $\{W(t), t \geq 0\}$ be the Brownian motion and $\{U(t), t \geq 0\}$ be the Ornstein-Uhlenbeck process. It is known that $\{U(t), t \geq 0\}$ and $\{W(e^t)/e^{t/2}, t \geq 0\}$ have the same distribution. Moreover

$$\mathbb{P}\Big(\sup_{0 \leq t \leq 1} W(t) \leq x\Big) = \mathbb{P}\Big(|W(1)| \leq x\Big) \sim (2/\pi)^{1/2} x$$

as $x \to 0$ and

$$\mathbb{P}\Big(\sup_{0 \leq t \leq T} U(t) \leq 0\Big) = \exp(-T/2 + o(T))$$

as $T \to \infty$.

Is there a connection between these two types of lower tail probabilities ?

Li and Shao (2003):

Let $\{Y_t, t \geq 0\}$ be an almost surely continuous stationary Gaussian process with $\mathbb{E}\, Y_t = 0$ and $\mathbb{E}\, Y_t^2 = 1$ for $t \geq 0$. Put $\rho(t) = \mathbb{E}\, Y_0 Y_t$. Assume that $\rho(t) \geq 0$. We have

(i) The limit

$$p(x) := \lim_{T \to \infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0 \leq t \leq T} Y_t \leq x\Big)$$

exists, left continuous, and

$$p(x) = \sup_{T>0} T^{-1} \ln \mathbb{P}\Big(\sup_{0 \le t \le T} Y_t \le x \Big)$$

for every $x \in \mathbb{R}^1$.

(ii) If $\rho(t)$ *is decreasing and*

$$a_{h,\theta}^2 := \inf_{0<t\le h} \frac{\rho(\theta t) - \rho(t)}{1-\rho(t)} > 0$$

for every $0 < h < \infty$ *and* $0 < \theta < 1$, *then* $p(x)$ *is continuous.*

To state the connection between lower tail probabilities of a non-stationary Gaussian process and its dual stationary Gaussian process, let $\{X_t, t \ge 0\}$ be a Gaussian process with $X_0 = 0$, $\mathbb{E}\, X_t = 0$. Assume that

(A1) $\mathbb{E}\, X_s X_t \ge 0$ and $\mathbb{E}\, X_t^2 = t^\alpha$ for $\alpha > 0$;

(A2) $\{Y_t = X(e^t)/e^{\alpha/2}, t \ge 0\}$ is a stationary Gaussian process;

(A3) $\{X_{at}, 0 \le t \le 1\}$ and $\{a^{\alpha/2} X_t, 0 \le t \le 1\}$ have the same distribution for each fixed $a > 0$.

(A4) $\rho(t) := \mathbb{E}\, Y_t Y_0$ is decreasing and condition () holds.

By subadditivity and the Slepian lemma,

$$c := -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} Y_t \le 0 \Big) = -\sup_{T>0} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} Y_t \le 0 \Big)$$

exists. Next result shows that the constant c is closely related to the rate of the lower tail probability $\mathbb{P}\Big(\sup_{0\le t\le 1} X_t \le x\Big)$.

- Li and Shao (2003):
  - Under conditions $(A1) - A(4)$, we have

    $$P(\sup_{0\le t\le 1} X_t \le x) = x^{2c_\alpha/\alpha + o(1)}$$

    as $x \longrightarrow 0$.
  - Let B_α be a fractional Brownian motion of order α $(0 < \alpha < 2)$ and put

    $$Y_\alpha(t) := \frac{B_\alpha(e^t)}{e^{t\alpha/2}}.$$

    Then

    $$c_\alpha = -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} Y_\alpha(t) \le 0\Big)$$

    exists. Moreover, $0 < c_\alpha < \infty$ and

    $$\mathbb{P}\Big(\sup_{0\le t\le 1} B_\alpha(t) \le x\Big) = x^{2c_\alpha/\alpha + o(1)} \quad \text{as} \quad x \longrightarrow 0$$

- Molchan (1999): $c_\alpha = 1 - \alpha/2$

Similarly, we have an alternative representation for the constant b in Example (c).

Let $Y(0) = 0$ and

$$Y(t) = \sqrt{2} t^2 \int_0^\infty W(u) e^{-ut} du$$

for $t > 0$, where W is the Brownian motion. Then $\mathbb{E}\, Y(t) = 0$ and

$$\mathbb{E}\, Y(t)Y(s) = \frac{2st}{s+t} \quad \text{for} \quad s, t > 0.$$

Hence $\{X_t\}$ in Example (c) and $\{Y(e^t)/e^{t/2}\}$ have the same distribution.

Li and Shao (2002):

We have

$$\mathbb{P}\Big(\sup_{0 \le t \le 1} Y(t) \le x \Big) = x^{b/2 + o(1)}$$

as $x \to 0$. *Furthermore,* $0.5 < b < 1$.

Open questions:

1. If $\{X_t, t \geq 0\}$ is a differentiable stationary Gaussian process with positive correlation, what is the limit

$$\lim_{T\to\infty} \frac{1}{T} \ln P\Big(\sup_{0\leq t\leq T} X_t \leq 0 \Big) \ ?$$

2. What is b?

3. Capture Time of the Fractional Brownian Motion Pursuit

Let $\{B_{k,\alpha}(t);\ t \geq 0\}(k = 0, 1, 2, \ldots, n)$ be independent fractional Brownian motions of order $\alpha \in (0, 2)$. Put

$$\tau_n := \tau_{n,\alpha} = \inf \left\{ t > 0 : \max_{1 \leq k \leq n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1 \right\}.$$

When is $\mathbb{E}(\tau_n)$ finite?

Note that

$$\begin{aligned} \mathbb{P}(\tau_n > s) &= \mathbb{P}\Big(\max_{1 \leq k \leq n} \sup_{0 \leq t \leq s} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < 1 \Big) \\ &= \mathbb{P}\Big(\max_{1 \leq k \leq n} \sup_{0 \leq t \leq 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < s^{-\alpha/2} \Big). \end{aligned}$$

Let

$$X_{k,\alpha}(t) = e^{-t\alpha/2} B_{k,\alpha}(e^t),\ k = 0, 1, \cdots, n$$

and

$$\gamma_{n,\alpha} := -\lim_{T \to \infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0 \leq t \leq T} \max_{1 \leq k \leq n} (X_{k,\alpha}(t) - X_{0,\alpha}(t)) \leq 0 \Big)$$

- Li and Shao (2003):

  $$\mathbb{P}\Big(\max_{1 \leq k \leq n} \sup_{0 \leq t \leq 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < x \Big) = x^{2\gamma_{n,\alpha}/\alpha + o(1)}$$

  as $x \to 0$

- Kesten (1992):

$$0 < \liminf_{n\to\infty} \gamma_{n,1}/\ln n \le \limsup_{n\to\infty} \gamma_{n,1}/\ln n \le 1/4$$

Conjecture: $\lim_{n\to\infty} \gamma_n/\ln n$ exists.

- Li and Shao (2002):

$$\frac{1}{d_\alpha} \le \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \le \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty,$$

where $d_\alpha = 2\int_0^\infty (e^{x\alpha} + e^{-x\alpha} - (e^x - e^{-x})^\alpha)dx$. In particular,

$$\lim_{n\to\infty} \frac{\gamma_n}{\ln n} = \frac{1}{4}$$

Conjecture:

$$\lim_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} = \frac{1}{d_\alpha}.$$

4. Some Comparison Inequalities

- Li and Shao (2002):

  Let $n \geq 3$, and let $(\xi_j, 1 \leq j \leq n)$ and $(\eta_j, 1 \leq j \leq n)$ be standard normal random variables with covariance matrices $R^1 = (r_{ij}^1)$ and $R^0 = (r_{ij}^0)$, respectively. Assume

  $$r_{ij}^1 \geq r_{ij}^0 \geq 0 \ \text{ for all } \ 1 \leq i, j \leq n$$

  Then

  $$\mathbb{P}\Big(\bigcap_{j=1}^n \{\eta_j \leq u_j\}\Big)$$
  $$\leq \mathbb{P}\Big(\bigcap_{j=1}^n \{\xi_j \leq u_j\}\Big) \leq \mathbb{P}\Big(\bigcap_{j=1}^n \{\eta_j \leq u_j\}\Big)$$
  $$\exp\Big\{ \sum_{1 \leq i < j \leq n} \ln\Big(\frac{\pi - 2\arcsin(r_{ij}^0)}{\pi - 2\arcsin(r_{ij}^1)}\Big) \exp\Big(- \frac{(u_i^2 + u_j^2)}{2(1 + r_{ij}^1)}\Big)\Big\}$$

  for any $u_i \geq 0, i = 1, 2, \cdots, n$ satisfying

  $$(r_{ki}^l - r_{ij}^l r_{kj}^l) u_i + (r_{kj}^l - r_{ij}^l r_{ki}^l) u_j \geq 0 \qquad (**)$$

  for $l = 0, 1$ and for all $1 \leq i, j, k \leq n$.

  Note: Condition (**) is satisfied if $u_i = u \geq 0$.

- Open question: Does the result remain valid without assuming (**)?
- Shao (2003):
  - Let $X_1, ..., X_n$ be jointly Gaussian random variables with mean zero. Then

$$P\Big(\max_{1\le i\le n}|X_i|\le x\Big) \ge 2^{-\min(k,n-k)/2}P\Big(\max_{1\le i\le k}|X_i|\le x\Big)P\Big(\max_{k<i\le n}|X_i|\le x\Big)$$

  - Let B_α be the fractional Brownian motion of order α. Then there exists $c_\alpha > 0$ such that

$$P\Big(\sup_{0\le s\le a}|B_\alpha(t)|\le x,\ \sup_{a\le t\le b}|B_\alpha(t)-B_\alpha(a)|\le y\Big) \ge c_\alpha P\Big(\sup_{0\le s\le a}|B_\alpha(t)|\le x\Big)P\Big(\sup_{a\le t\le b}|B_\alpha(t)-B_\alpha(a)|\le y\Big)$$

    for any $0 < a < b$, $x > 0$ and $y > 0$.
  - Assume $\boldsymbol{X} = (X_1, ..., X_n)' \sim N(\mathbf{0}, \mathbf{\Sigma_1})$, and $\boldsymbol{Y} = (Y_1, ..., Y_n)' \sim N(\mathbf{0}, \mathbf{\Sigma_2})$. If $\mathbf{\Sigma_2} - \mathbf{\Sigma_1}$ is positive semidefinite, then

$$\forall\, C \subset R^n,\ P(\boldsymbol{Y} \in C) \ge (|\mathbf{\Sigma_1}|/|\mathbf{\Sigma_2}|)^{\mathbf{1/2}}\mathbf{P}(\boldsymbol{X} \in \mathbf{C}).$$

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