Heat Kernels,

Symplectic Geometry,

Moduli Spaces and Finite Groups

Basic Idea: Heat Kernel as a unifying technique to treat problems of different nature.

We will use the heat kernel of the simplest operator: The Laplacian.

Heat kernel in R^n:

$$\Delta = \sum \frac{\partial^2}{\partial {x^i}^2},$$

Fundamental solution of

$$(\frac{\partial}{\partial t} - \Delta)H = 0$$

$$H(t, x, x_0) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x - x_0|^2}{4t}).$$

On general (compact) manifold M,

$$H(t, x, x_0) =$$

$$\frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x - x_0|^2}{4t})\{a_0 + a_1 t + \cdots\}.$$

Fundamental solution of

$$(\frac{\partial}{\partial t} - \Delta)H = 0$$

with Laplacian Δ of given metric.

Localizing property:

$$H(t, x, x_0) \xrightarrow{t \to 0} \delta(x - x_0).$$

On the other hand:

$H(t, x, x_0)$ also has global expression in terms of eigen-functions.

Special cases: Theta-functions and modular transformation.

(Famous applications: Mckean-Singer: Atiyah-Singer, Atiyah-Bott, equivariant localization formulas $\cdots$.)

Serge Lang: Heat kernel is everything!?

Simple idea: Consider

$$f: \ M \to N$$

smooth map between smooth manifolds.

Let $H^N(t, x, x_0)$ be the heat kernel on N.

Consider the integral:

$$I(t) = \int_M H^N(t, f(y), x_0) dy$$

dy: a measure on M.

As $t \to 0$,

$$I(t) \longrightarrow \int_{N_\delta(f^{-1}(x_0))} H^N(t, f(y), x_0)dy$$

from localizing property.

Then compute $I(t)$ globally on M or N.

Principle: Local $\Leftrightarrow$ Global.

Heat kernel as a bridge.

Will discuss four applications:

(1) Witten's Nonabelian localization formula in symplectic geometry. (Wu, J-K).

(2) Intersection numbers on moduli space of flat G-bundles on a Riemann surface: Witten's formulas; Verlinde formula (Bismut-L).

(3) Measures of the solution moduli for equations in compact Lie groups: (Diaconis.)

(4) Numbers of solutions of equations in finite groups. (Freed-Q, Serre).

Other applications: (a) study fundamental groups of 3-manifolds; (b) group-valued moment maps; (c) hyper-kahler moment maps.

Warm-up:

(1) M, compact symplectic manifold, K compact Lie group, k its Lie algebra, k^* the dual, and $< \, , \, >$ the metric induced from Killing form.

Let ω be the symplectic form on M. Assume K acts on M, preserving ω, with

$$\mu : \; M \longrightarrow k^*$$

the moment map. For $X \in k$,

$$d(\mu, X) = i_{X_M}\omega,$$

X_M: the induced vector field on M.

(μ, X): pairing on $k^* \times k$.

The symplectic quotient

$$M_K = \mu^{-1}(0)/K$$

(= GIT quotient in projective category by $K_{\mathbb{C}}$.)

In this case,

$$N = k^* \simeq \mathbb{R}^n$$

and

$$H^N(t, x, x_0) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x - x_0|^2}{4t}).$$

Consider the integral

$$I(t) = \int_M H(t, \mu(y), 0) e^{\omega},$$

where symplectic volume

$$e^{\omega} = \frac{\omega^m}{m!}$$

with $m = \dim_{\mathbb{C}} M$.

(i) Local computation: $t \to 0$ (Guillemin-S):

$$I(t) = \int_{M_K} e^{\omega_0 - t<F,F>} + O(e^{-\delta^2/4t}),$$

(Computation in δ-neighborhood of $\mu^{-1}(0)$.

ω_0: the induced symplectic form on M_K,

F: the curvature of $\pi: \ \mu^{-1}(0) \xrightarrow{K} M_K$.

(ii) Global computation: rewrite (Fourier transform),

$$I(t) = \int_k e^{-t<\varphi,\varphi>} \int_M e^{\omega+i(\mu,\varphi)} d\varphi.$$

Reduce to the fixed points in M of the maximal torus.

Take $K = S^1$ as example. $\omega+i(\mu,\varphi)$ the equivariant symplectic form.

Atiyah-Bott Localization formula

$$\int_M e^{\omega+i(\mu,\varphi)} = \sum_F \int_F \frac{i_F^* e^{\omega+i(\mu,\varphi)}}{e_T(N_{F/M})}.$$

$\{F\}$: fixed components; i_F^*: restriction.

$e_T(N_{F/M})$: the equivariant Euler class of the normal bundle of F in M.

Final formula: Witten's nonabelian localization,

$$I(t) = \int_{M_K} e^{\omega_0 - t<F,F>} + O(e^{\delta^2/4t})$$

$$= \int_k e^{-t<\varphi,\varphi>} d\varphi \sum_F \int_F \frac{i_F^* e^{\omega + i(\mu,\varphi)}}{e_T(N_{F/M})}.$$

for $K = S^1$, (S. Wu). Or to maximal torus (J-K).

Take limit $t \to 0$.

Expand in t-polynomial, compare coefficients.

(2) Intersection numbers on moduli space of flat G-bundles on a Riemann surface.

Consider map:

$$f: \quad G^{2g} \times O_c \to G$$

with

$$f(x_1, \cdots, y_g; z) = \prod_{j=1}^{g} [x_j, y_j] z.$$

General cases are the same.

O_c: conjugacy class through (generic) $c \in G$.

G with the bi-invariant metric induced by the Killing form.

Heat kernel on G:

$$H(t,x,y) = \frac{1}{|G|} \sum_{\lambda \in P_+} d_\lambda \cdot \chi_\lambda(xy^{-1}) e^{-tp_c(\lambda)}$$

$|G|$: volume of G

P_+: all irreducible representations, identified as a lattice in t^*, dual of the Lie algebra of the maximal torus $T \subset G$.

χ_λ: the character of λ.

d_λ: the dimension of λ.

$p_c(\lambda) = |\lambda + \rho|^2 - |\rho|^2$. $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Δ^+: positive roots.

Moduli space

$$\mathcal{M}_c = f^{-1}(e)/G.$$

G acts on $G^{2g} \times O_c$ by the conjugation γ:

$$\gamma: \; G \to G^{2g} \times O_c$$

with

$$\gamma(w)(x_1, \cdots, y_g; z) = (wx_1w^{-1}, \cdots, wy_gw^{-1}; wzw^{-1}).$$

Consider the integral:

$$I(t) = \int_{h \in G^{2g} \times O_c} H(t, f(h), e)dh$$

dh: induced measure on $G^{2g} \times O_c$.

(i) Local computation: $t \to 0$,

$$I(t) = \frac{|G|}{|Z(G)|} \int_{\mathcal{M}_c} d\nu_c + O(e^{-\delta^2/4t})$$

where $d\nu_c$ is the Reidemeister torsion $\tau(\mathcal{C}'_c)$ of the complex

$$\mathcal{C}'_c : \quad 0 \to g \xrightarrow{d\gamma} g^{2g} \oplus T_c O_c \xrightarrow{df} g \to 0$$

with $g \simeq T_e G$. (Forman)

Deformation complex $\Longrightarrow$

$$T\mathcal{M}_c \simeq H^1(\mathcal{C}'_c).$$

Poincare duality (Witten, B-L, Milnor, Johnson) $\Longrightarrow$

$$d\nu_c = \tau(\mathcal{C}'_c) = (2\pi)^{2N_c} |j(c)| \frac{\omega_c^{N_c}}{N_c!}$$

with

$$|j(c)| = \ \det(I - \mathrm{Ad}(c))|^{\frac{1}{2}}$$

on $T_c O_c$: Weyl denominator; R-torsion of boundary.

ω_c: the natural symplectic structure on $\mathcal{M}_c$, induced from Poincare duality on the Riemann surface.

(ii) Global computation: The character relations:

$$\int_G \chi_\lambda(wyzy^{-1}z^{-1})dz = \frac{|G|}{d_\lambda} \chi_\lambda(wy)\chi_\lambda(y^{-1}),$$

$$\int_G \chi_\lambda(wy)\chi_\lambda(y^{-1})dy = \frac{|G|}{d_\lambda}\chi_\lambda(w)$$

and

$$\int_{O_c} h(g)dv_g = \frac{|j(c)|^2}{|Z_c|}\int_G h(gcg^{-1})dg$$

for any continuous function h on O_c.

$Z_c \simeq t$, Lie algebra of the centralizer of (generic) c.

Summarize:

$$\int_{\mathcal{M}_c} e^{\omega_c} = |Z(G)| \frac{|G|^{2g-1}|j(c)|}{(2\pi)^{2N_c}|Z_c|} \sum_{\lambda \in P_+} \frac{\chi_\lambda(c)}{d_\lambda^{2g-1}} e^{-tp_c(\lambda)}$$

$$+O(e^{-\delta^2/4t}).$$

For $u \in Z(G)$ in center, write $c = u \exp C$ near u. $C \in t$.

A little bit of symplectic geometry applied to the fibration:

$$G/T \to \mathcal{M}_c \xrightarrow{\pi} \mathcal{M}_u$$

(Assume $\mathcal{M}_u$ smooth):

$$\omega_c = \pi^* \omega_u + \nu_c$$

ν_c: the symplectic structure on fibers.

Take derivatives with respect to C, and take limit $c \to u$:

$\int_{\mathcal{M}_u} p(\sqrt{-1}\Omega) e^{\omega_u} = |Z(G)| \frac{|G|^{2g-2}}{(2\pi)^{2N_u}}$.

$$\lim_{c \to u} \lim_{t \to 0} \sum_{\lambda \in P_+} \frac{\chi_\lambda(c)}{d_\lambda^{2g-1}} p(\lambda + \rho) e^{-t p_c(\lambda)}.$$

p: any Weyl-invariant polynomial.

$2\pi\Omega$: curvature form of $f^{-1}(e) \to \mathcal{M}_u$

Derivative + Heat kernel $\Longrightarrow$ symplectic volume:

$$\mathbf{Vol}(\mathcal{M}_c) = \int_{\mathcal{M}_c} e^{\omega_c}$$

is a polynomial in C of degree at most $2g\,|\Delta^+|$ (piecewise):

If $\deg p \geq 2g|\Delta^+|$,

$$\int_{\mathcal{M}_u} p(\sqrt{-1}\Omega)e^{\omega_u} = 0.$$

(Newstead conjecture for $G = SU(2)$, Atiyah-Bott, Donaldson, Kirwan, Zagier. Witten vanishing for $SU(n)$. Gieseker's vanishing for Chern classes.)

Remarks: The integrals

$$\int_{\mathcal{M}_u} p(\sqrt{-1}\Omega)e^{\omega_u}$$

contains all the information for Verlinde formula, since

$$\dim H^0(\mathcal{M}_u, L^k) = \int_{\mathcal{M}_u} \hat{A}\sqrt{-1}\Omega)e^{N_k\omega_u}$$

with $c_1(L) = \omega_u$, $k >> 0$. (AS index formula).

Bismut-Labourie: Rewrite infinite sum as "finite sum": residues.

Derivatives of Volume + residues $\Longrightarrow$ Verlinde.

($G = SU(n)$ Szenes' residues; general G, orbifold singularities. More punctures.)

Products of Lie groups + Heat Kernel $\Longrightarrow$ geometry of moduli spaces!

In general, one may consider

$$I(t) = \int_{G^{2g}\times O_c} F(h)H(t, f(h), e)dh$$

for G-invariant function $F(h)$.

$$I(t) \Longrightarrow \int_{\mathcal{M}_c} \bar{F}(h)e^{\omega_c}.$$

Heat kernel method = finite dimensional analogue of Witten's path integral approach:

$G^{2g} \leftrightarrow \mathcal{A}$, the connection space.

(3) Motivated by a conjecture of Diaconis.

Consider the induced measure on the solution space of

$$f_j(x_1, \cdots, x_m) = c_j \in G, \;\; j = 1, 2, \cdots, n$$

in compact Lie group G^m.

This gives a map

$$f = (f_1, \cdots f_n) : \;\; G^m \longrightarrow G^n.$$

The heat kernel integral

$$I(t) = \int_{G^n} \prod_{j=1}^{n} H(t, f_j(h), c_j) dh$$

gives the answer immediately.

Example: $\{H_j\}$ subgroups of G. Consider the equation

$$\prod_{j=1}^{n} x_j u_j x_j^{-1} = x$$

in $G^n \times \prod_j H_j$.

Consider map:

$$f: \quad G^n \times \prod_{j=1}^{n} H_j \to G$$

$$f(x_1, \cdots, x_n; u_1, \cdots, u_n) = \prod_j x_j u_j x_j^{-1}.$$

Consider the integral

$$I(t) = \int_{h \in G^n \times \prod_j H_j} H(t, f(h), x) dh$$

$$= \int_G H(t, y, x) F(y) dy.$$

Local + global computations:

$$f_* dh(x) = |G|^{n-1} \sum_{\lambda \in P_+} \frac{\prod_j \int_{H_j} \chi_\lambda(u_j) du_j}{d_\lambda^{n-2}} \chi_\lambda(x^{-1}) dx.$$

dh: biinvariant measure on $G^n \times \prod_{j=1}^n H_j$,

dx: biinvariant measure on G.

Example: Find the measure for the solution space: n-commutator equation,

$$[x_1, [x_2, [\cdots, x_n]]] = x$$

in G^n. (Induction formula).

More $\cdots$.

(4) Count solutions in finite groups.

G finite group, its heat kernel is

$$H(t,x,y) = \frac{1}{|G|} \sum_{\lambda \in P_+} d_\lambda \cdot \chi_\lambda(xy^{-1}) e^{-tp_c(\lambda)}$$

$|G|$: number of elements in G,

P_+: all irreducible representations,

$p_c(\lambda)$: a function on P_+.

Same method for compact Lie groups works well: Replace integrals by sums over G.

Example: Solve equation

$$\prod_{j=1}^{n} [x_j, y_j] \prod_{j=1}^{n} z_j = e$$

in $G^{2g} \times \prod_j O_{c_j}$, with O_{c_j} conjugacy class of $c_j \in G$.

Consider map

$$f(x_1, y_1, \cdots, x_g, y_g; z_1, \cdots, z_n) = \prod_{j=1}^{n} [x_j, y_j] \prod_{j=1}^{n} z_j,$$

and integral (sum):

$$I(t) = \int_{G^{2g} \times \prod_j O_{c_j}} H(t, f(h), e) dh.$$

Local + global computations give the number of solutions:

$$S_{g,n} = \frac{|G|^{2g+n-1}}{\prod_{j=1}^{n} |Z_{c_j}|} \sum_{\lambda \in P_+} \frac{\prod_{j=1}^{n} \chi_\lambda(c_j)}{d_\lambda^{2g+n-2}},$$

Z_{c_j} the centralizer of c_j. Character formulas used.

$S_{g,n}$ integer(?): $\chi_\lambda(c_j)$ is algebraic integer.

$S_{g,n}$ known: Freed-Quinn, $n = 0$; Serre.

Strunkov: $S_{g,n} \Longrightarrow$ Brauer p-block conjecture.

Examples: Formulas for numbers:

(a) In G^n: $[x_1, [x_2, \cdots, x_{n-1}], x_n] = e$

(b) In $G^n \times \prod_{j=1}^{n} H_j$: $\prod_j x_j u_j x_j^{-1} = e$.

(5) For a two or three manifold M with a G-bundle P on it, simplicial decompositions always induce certain equations in G:

Presentations of $\pi_1(M) \Longrightarrow$ Equations in G.

(6) Group-valued moment maps: $\mu : \ M \to G$. Hyper-Kahler moment maps....