GEOMETRIC ASPECTS OF THE MODULI SPACE OF RIEMANN SURFACES

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Contents

1. Introduction

The study of moduli space and Teichmüller space has a long history. These two spaces lie in the intersections of the researches in many areas of mathematics and physics. Many deep results have been obtained in history by many famous mathematicians. Here we will only mention a few that are closely related to our discussions.

Riemann was the first who considered the space $\mathcal{M}$ of all complex structures on an orientable surface modulo the action of orientation preserving diffeomorphisms. He derived the dimension of this space

$$\dim_{\mathbb{R}} \mathcal{M} = 6g - 6$$

where $g \geq 2$ is the genus of the topological surface.

In 1940's, Teichmüller considered a cover of $\mathcal{M}$ by taking the quotient of all complex structures by those orientation preserving diffeomorphims which are isotopic to the identity map. The Teichmüller space $\mathcal{T}_g$ is a contractible set in $\mathbb{C}^{3g-3}$. Furthermore, it is a pseudoconvex domain. Teichmüller also introduced the Teichmüller metric by first taking the L^1 norm on the cotangent space of $\mathcal{T}_g$ and then taking the dual norm on the tangent space. This is a Finsler metric. Two other interesting Finsler metrics are the Carathéodory metric and the Kobayashi metric. These Finsler metrics have been powerful tools to study the hyperbolic property of the moduli and the Teichmüller spaces and the mapping class groups. For example in 1970's Royden proved that the Teichmüller metric and the Kobayashi metric are the same, and as a corollary he proved the famous result that the holomorphic automorphism group of the Teichmüller space is exactly the mapping class group.

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Based on the Petersson pairing on the spaces of automorphic forms, Weil introduced the first Hermitian metric on the Teichmüller space, the Weil-Petersson metric. It was shown by Ahlfors that the Weil-Petersson metric is Kähler and its holomorphic sectional curvature is negative. The works of Ahlfors and Bers on the solutions of Beltrami equation put a solid fundation of the theory of Teichmüller space and moduli space [1]. Wolpert studied in details the Weil-Petersson metric including the precise upper bound of its Ricci and holomorphic sectional curvature. From these one can derive interesting applications in algebraic geometry. For example, see [9] .

Moduli spaces of Riemann surfaces have also been studied in details in algebraic geometry since 1960. The major tool is the geometric invariant theory developed by Mumford. In 1970's, Deligne and Mumford studied the projective property of the moduli space and they showed that the moduli space is quasi-projective and can be compactified naturally by adding in the stable nodal surfaces [3]. Fundamental works have been done by Gieseker, Harris and many other algebraic geometers.

The work of Cheng-Yau [2] in the early 80s showed that there is a unique complete Kähler-Einstein metric on the Teichmüller space and is invariant under the moduli group action. Thus it descends to the moduli space. As it is well-known, the existence of the Kähler-Einstein metric gives deep algebraic geometric results, so it is natural to understand its properties like the curvature and the behaviors near the compactification divisor. In the early 80s, Yau conjectured that the Kähler-Einstein metric is equivalent to the Teichmüller metric and the Bergman metric [2], [24], [14].

In 2000, McMullen introduced a new metric, the McMullen metric by perturbing the Weil-Petersson metric to get a complete Kähler metric which is complete and Kähler hyperbolic. Thus the lowest eigenvalue of the Laplace operator is positive and the L^2-cohomology is trivial except for the middle dimension [13].

The moduli space appears in many subjects of mathematics, from geometry, topology, algebraic geometry to number theory. For example, Faltings' proof of the Mordell conjecture depends heavily on the moduli space which can be defined over the integer ring. Moduli space also appears in many areas of theoretical physics. In string theory, many computations of path integrals are reduced to integrals of Chern classes on the moduli space. Based on conjectural physical theories, physicists have made several amazing conjectures about generating series of Hodge integrals for all genera and all marked points on the moduli spaces. The proofs of these conjectures supply strong evidences to their theories.

Our goal of this project is to understand the geometry of the moduli spaces. More precisely, we want to understand the relationships among all of the known canonical complete metrics introduced in history on the moduli and the Teichmüller spaces, and more importantly to introduce new complete Kähler metrics with good curvature properties: the Ricci metric and the perturbed Ricci metric. Through a detailed study we proved that these new metrics have very good curvature properties and very nice Poincaré-type asymptotic behaviors [10], [11]. In particular we proved that the perturbed Ricci metric has bounded negative Ricci and holomorphic sectional curvature and has bounded geometry. To the knowledge of the authors this is the first known such metric on moduli space and the Teichmüller spaces with such good properties. We know that the Weil-Petersson metric has negative Ricci and holomorphic sectional curvature, but it is incomplete and its curvatures are not bounded from below. Also note that one has no control on the signs of the curvatures of the other complete Kähler metrics mentioned above.

We have obtained a series of results. In [10] and [11] we have proved that all of these known complete metrics are actually equivalent, as consequences we proved two old conjectures of Yau about the equivalence between the Kähler-Einstein metric and the Teichmüller metric and also its equivalence with the Bergman metric. In both [24] and [14] which were both written in early 80s, Yau raised various questions about the Kähler-Einstein metric on the Teichmüller space. By using the curvature properties of these new metrics, we obtained good understanding of the

Kähler-Einstein metric such as its boundary behavior and the strongly bounded geometry. As one consequence we proved the stability of the logarithmic extension of the cotangent bundle of the moduli space [11]. Note that the major parts of our papers were to understand the Kähler-Einstein metrics and the two new metrics. One of our goal is to find a good metric with the best possible curvature property. The perturbed Ricci metric is close to be such metric. We hope to understand its Riemannian curvature in the future. The most difficult part of our results is the study of the curvature properties and the asymptotic behaviors of the new metrics near the boundary, only from which we can derive geometric applications such as the stability of the logarithmic cotangent bundle. The comparisons of those classical metrics as well as the two new metrics are quite easy and actually simple corollaries of the study and the basic definitions of those metrics.

Our first paper was post in the webpage since February 2004 and widely circulated. Since then the first and the second author have given several lectures about the main results and key ideas of both of our papers. In July we learned from the announcement of S.-K. Yeung in Hong Kong University [1] where he announced he could prove a small and easy part of our results about the equivalences of some of these metrics by using a bounded pluri-subharmonic function. [2]

The purpose of this note is to give a brief overview of our results and their background. It is based on the lecture delivered by the first author in the First International Conference of Several Complex Variables held in the Capital Normal University in August 23-28, 2004. All of the main results mentioned here are contained in [10] and [11] which the interested reader may read for details. They have been circulated for a while. The first author would like to thank the organizers for their invitation and hospitality.

2. The Topological Aspects of the Moduli Space

The topology of the Teichmüller space is trivial, since it is topologically a ball. But how to compactify it in a natural and useful way is still an interesting problem. Penner has done important works on this problem. The compactification of Teichmüller space is useful in three dimensional topology. The topology of the moduli space and its compactification is highly nontrivial and have been well-studied for the past years from many point of views. Here we only mention the recently proved Mumford conjecture about the stable cohomology of the moduli spaces; the Witten conjecture about the KdV equations for the generating series of the integrals of the ψ classes; the Mariño-Vafa conjecture about the closed expressions for the generating series of triple Hodge integrals.

The first two results mentioned above are already well-known. Here we would like to explain a little more details about the Mariño-Vafa conjecture proved in [8] which gives a closed formula for the generating series of triple Hodge integrals of all genera and all possible marked points, in terms of Chern-Simons knot invariants.

Hodge integrals are defined as the intersection numbers of λ classes and ψ classes on the Deligne-Mumford moduli spaces of stable Riemann surfaces $\overline{\mathcal{M}}_{g,h}$, the moduli with h marked points. Recall that a point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) Riemann surface C and h smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are the Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

[1] http://hkumath.hku.hk/∼imr/records0304/GEO-YeungSK.pdf

[2] We received a hard copy of Yeung's paper in November 2004 where he used a similar method to ours in [11] to compare the Bergman, the Kobayashi and the Carathéodory metric. It should be interesting to see how one can use the bounded psh function to derive these equivalences.

On the other hand the cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h};\mathbb{Q}).$$

Let us define

$$\Lambda^\vee_g(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

The Mariño-Vafa conjecture states that the generating series of the triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(\tau)\Lambda^\vee_g(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

for all g and all h can be expressed by close formulas of finite expression in terms of the representations of symmetric groups, or the Chern-Simons knot invariants. Here τ is a parameter and μ_i are some integers. Many interesting Hodge integral identities can be easily derived from this formula.

The Mariño-Vafa conjecture originated from the large N duality between the Chern-Simons and string theory. It was proved by exploring differential equations from both geometry and combinatorics. The interested reader may read [8] for more details.

3. The Background of the Teichmüller Theory

In this section, we recall some basic facts in Teichmüller theory and introduce various notations for the following discussions. Please see [4] and [18] for more details.

Let Σ be an orientable surface with genus $g \geq 2$. A complex structure on Σ is a covering of Σ by charts such that the transition functions are holomorphic. By the uniformization theorem, if we put a complex structure on Σ, then it can be viewed as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a Fuchsian group. Thus there is a unique Kähler-Einstein metric, or the hyperbolic metric on Σ.

Let $\mathcal{C}$ be the set of all complex structures on Σ. Let $Diff^+(\Sigma)$ be the group of orientation preserving diffeomorphisms and let $Diff^+_0(\Sigma)$ be the subgroup of $Diff^+(\Sigma)$ consisting of those elements which are isotopic to identity.

The groups $Diff^+(\Sigma)$ and $Diff^+_0(\Sigma)$ act naturally on the space $\mathcal{C}$ by pull-back. The Teichmüller space is a quotient of the space $\mathcal{C}$

$$\mathcal{T}_g = \mathcal{C}/Diff^+_0(\Sigma).$$

From the famous Bers embedding theorem, now we know that $\mathcal{T}_g$ can be embedded into $\mathbb{C}^{3g-3}$ as a pseudoconvex domain and is contractible. Let

$$\mathrm{Mod}_g = Diff^+(\Sigma)/Diff^+_0(\Sigma)$$

be the group of isotopic classes of diffeomorphisms. This group is called the (Teichmüller) moduli group or the mapping class group. Its representations are of great interests in topology and in quantum field theory.

The moduli space $\mathcal{M}_g$ is the space of distinct complex structures on Σ and is defined to be

$$\mathcal{M}_g = \mathcal{C}/Diff^+(\Sigma) = \mathcal{T}_g/\mathrm{Mod}_g.$$

The moduli space is a complex orbifold.

For any point $s \in \mathcal{M}_g$, let $X = X_s$ be a representative of the corresponding class of Riemann surfaces. By the Kodaira-Spencer deformation theory and the Hodge theory, we have

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T^*_X\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

Pick $\mu \in HB(X)$ and $\varphi \in Q(X)$. If we fix a holomorphic local coordinate z on X, we can write $\mu = \mu(z)\frac{\partial}{\partial z} \otimes d\bar{z}$ and $\varphi = \varphi(z)dz^2$. Thus the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \varphi] = \int_X \mu(z)\varphi(z)dzd\bar{z}.$$

By the Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} HB(X) = \dim_{\mathbb{C}} Q(X) = 3g-3$$

which implies

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g-3.$$

4. Metrics on the Teichmüller Space and the Moduli Space

There are many very famous classical metrics on the Teichmüller and the moduli spaces and they have been studied independently by many famous mathematicians. Each metric has played important role in the study of the geometry and topology of the moduli and Teichmüller spaces.

There are three Finsler metrics: the Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric $\|\cdot\|_K$ and the Carathéodory metric $\|\cdot\|_C$. They are all complete metrics on the Teichmüller space and are invariant under the moduli group action. Thus they descend down to the moduli space as complete Finsler metrics.

There are seven Kähler metrics: the Weil-Petersson metric ω_{WP} which is incomplete, the Cheng-Yau's Kähler-Einstein metric ω_{KE}, the McMullen metric ω_C, the Bergman metric ω_B, the asymptotic Poincaré metric on the moduli space ω_P, the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\tilde{\tau}}$. The last six metrics are complete. The last two metrics are new metrics studied in details in [10] and [11].

Now let us give the precise definitions of these metrics and state their basic properties.

The Teichmüller metric was first introduced by Teichmüller as the L^1 norm in the cotangent space. For each $\varphi = \varphi(z)dz^2 \in Q(X) \cong T_X^*\mathcal{M}_g$, the Teichmüller norm of φ is

$$\|\varphi\|_T = \int_X |\varphi(z)| \; dzd\bar{z}.$$

By using the duality, for each $\mu \in HB(X) \cong T_X\mathcal{M}_g$,

$$\|\mu\|_T = \sup\{Re[\mu;\varphi] \mid \|\varphi\|_T = 1\}.$$

Please see [4] for details. It is known that Teichmüller metric has constant holomorphic sectional curvature -1.

The Kobayashi and the Carathéodory metrics can be defined for any complex space in the following way: Let Y be a complex manifold and of dimension n. let Δ_R be the disk in $\mathbb{C}$ with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincaré metric on Δ. Let $p \in Y$ be a point and let $v \in T_pY$ be a holomorphic tangent vector. Let $\mathrm{Hol}(Y,\Delta_R)$ and $\mathrm{Hol}(\Delta_R, Y)$ be the spaces of holomorphic maps from Y to Δ_R and from Δ_R to Y respectively. The Carathéodory norm of the vector v is defined to be

$$\|v\|_C = \sup_{f \in \mathrm{Hol}(Y,\Delta)} \|f_* v\|_{\Delta,\rho}$$

and the Kobayashi norm of v is defined to be

$$\|v\|_K = \inf_{f \in \mathrm{Hol}(\Delta_R,Y),\ f(0)=p,\ f'(0)=v} \frac{2}{R}.$$

The Bergman (pseudo) metric can also be defined for any complex space Y provided the Bergman kernel is positive. Let K_Y be the canonical bundle of Y and let W be the space of L^2

holomorphic sections of K_Y in the sense that if $\sigma \in W$, then

$$\|\sigma\|_{L^2}^2 = \int_Y (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma} < \infty.$$

The inner product on W is defined to be

$$(\sigma, \rho) = \int_Y (\sqrt{-1})^{n^2} \sigma \wedge \overline{\rho}$$

for all $\sigma, \rho \in W$. Let $\sigma_1, \sigma_2, \cdots$ be an orthonormal basis of W. The Bergman kernel form is the non-negative (n,n)-form

$$B_Y = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} \sigma_j \wedge \overline{\sigma}_j.$$

With a choice of local coordinates $z_i, \cdots, z_n$, we have

$$B_Y = BE_Y(z, \overline{z})(\sqrt{-1})^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

where $BE_Y(z, \overline{z})$ is called the Bergman kernel function. If the Bergman kernel B_Y is positive, one can define the Bergman metric

$$B_{i\overline{j}} = \frac{\partial^2 \log BE_Y(z, \overline{z})}{\partial z_i \partial \overline{z}_j}.$$

The Bergman metric is well-defined and is nondegenerate if the elements in W separate points and the first jet of Y. In this case, the Bergman metric is a Kähler metric.

Remark 4.1. Both the Teichmüller space and the moduli space are equipped with the Bergman metrics. However, the Bergman metric on the moduli space is different from the metric induced from the Bergman metric of the Teichmüller space. The Bergman metric defined on the moduli space is incomplete due to the fact that the moduli space is quasi-projective and any L^2 holomorphic section of the canonical bundle can be extended over. However, the induced one is complete as we shall see later.

The basic properties of the Kobayashi, the Carathéodory and the Bergman metrics are stated in the following proposition. Please see [7] for the details.

Proposition 4.1. *Let X be a complex space. Then*

(1) $\|\cdot\|_{C,X} \leq \|\cdot\|_{K,X}$;
(2) *Let Y be another complex space and $f : X \to Y$ be a holomorphic map. Let $p \in X$ and $v \in T_pX$. Then $\|f_*(v)\|_{C,Y,f(p)} \leq \|v\|_{C,X,p}$ and $\|f_*(v)\|_{K,Y,f(p)} \leq \|v\|_{K,X,p}$;*
(3) *If $X \subset Y$ is a connected open subset and $z \in X$ is a point. Then with any local coordinates we have $BE_Y(z) \leq BE_X(z)$;*
(4) *If the Bergman kernel is positive, then at each point $z \in X$, a peak section σ at z exists. Such a peak section is unique up to a constant factor c with norm 1. Furthermore, with any choice of local coordinates, we have $BE_X(z) = |\sigma(z)|^2$;*
(5) *If the Bergman kernel of X is positive, then $\|\cdot\|_{C,X} \leq 2\|\cdot\|_{B,X}$;*
(6) *If X is a bounded convex domain in $\mathbb{C}^n$, then $\|\cdot\|_{C,X} = \|\cdot\|_{K,X}$;*
(7) *Let $|\cdot|$ be the Euclidean norm and let B_r be the open ball with center 0 and radius r in $\mathbb{C}^n$. Then for any holomorphic tangent vector v at 0,*

$$\|v\|_{C,B_r,0} = \|v\|_{K,B_r,0} = \frac{2}{r}|v|$$

where $|v|$ is the Euclidean norm of v.

The three Finsler metrics have been very powerful tools in understanding the hyperbolic geometry of the moduli spaces, and the mapping class group. It is also known since 70's that the Bergman metric on the Teichmüller space is complete.

The Weil-Petersson metric is the first Kähler metric defined on the Teichmüller and the moduli space. It is defined by using the L^2 inner product on the tangent space in the following way: Let $\mu, \nu \in T_X\mathcal{M}_g$ be two tangent vectors and let λ be the unique Kähler-Einstein metric on X. Then the Weil-Pertersson metric is

$$h(\mu, \nu) = \int_X \mu\overline{\nu}\, dv$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form. Details can be found in [10], [12] and [21].

The curvatures of the Weil-Petersson metric have been well-understood due to the works of Ahlfors, Royden and Wolpert. Its Ricci and holomorphic sectional curvature are all negative with negative upper bound, but with no lower bound. Its boundary behavior is understood, from which it is not hard to see that it is an incomplete metric.

The existence of the Kähler-Einstein metric was given by the work of Cheng-Yau [2]. Its Ricci curvature is -1. Namely,

$$\partial\overline{\partial}\log\omega_{_{KE}}^n = \omega_{_{KE}}$$

where $n = 3g - 3$. They actually proved that a bounded domain in $\mathbb{C}^n$ admits a complete Kähler-Einstein metric if and only if it is pseudoconvex.

The McMullen $1/l$ metric defined in [13] is a perturbation of the Weil-Petersson metric by adding a Kähler form whose potential involves the short geodesic length functions on the Riemann surfaces. For each simple closed curve γ in X, let $l_\gamma(X)$ be the length of the unique geodesic in the homotopy class of γ with respect to the unique Kähler-Einstein metric. Then the McMullen metric is defined as

$$\omega_{1/l} = \omega_{_{WP}} - i\delta \sum_{l_\gamma(X)<\epsilon} \partial\overline{\partial}\mathrm{Log}\frac{\epsilon}{l_\gamma}$$

where ϵ and δ are small positive constants and $\mathrm{Log}(x)$ is a smooth function defined as

$$\mathrm{Log}(x) = \begin{cases} \log x & x \geq 2 \\ 0 & x \leq 1. \end{cases}$$

This metric is Kähler hyperbolic which means it satisfies the following conditions:

(1) $(\mathcal{M}_g, \omega_{1/l})$ has finite volume;
(2) The sectional curvature of $(\mathcal{M}_g, \omega_{1/l})$ is bounded above and below;
(3) The injectivity radius of $(\mathcal{T}_g, \omega_{1/l})$ is bounded below;
(4) On $\mathcal{T}_g$, the Kähler form $\omega_{1/l}$ can be written as $\omega_{1/l} = d\alpha$ where α is a bounded 1-form.

An immediate consequence of the Kähler hyperbolicity is that the L^2-cohomology is trivial except for the middle dimension.

The asymptotic Poincaré metric can be defined as a complete Kähler metric on a complex manifold M which is obtained by removing a divisor Y with only normal crossings from a compact Kähler manifold $(\overline{M}, \omega)$.

Let $\overline{M}$ be a compact Kähler manifold of dimension m. Let $Y \subset \overline{M}$ be a divisor of normal crossings and let $M = \overline{M} \setminus Y$. Cover $\overline{M}$ by coordinate charts $U_1, \cdots, U_p, \cdots, U_q$ such that $(\overline{U}_{p+1} \cup \cdots \cup \overline{U}_q) \cap Y = \Phi$. We also assume that for each $1 \leq \alpha \leq p$, there is a constant n_α such that $U_\alpha \setminus Y = (\Delta^*)^{n_\alpha} \times \Delta^{m-n_\alpha}$ and on U_α, Y is given by $z_1^\alpha \cdots z_{n_\alpha}^\alpha = 0$. Here Δ is the disk of radius $\frac{1}{2}$ and Δ^* is the punctured disk of radius $\frac{1}{2}$. Let $\{\eta_i\}_{1\leq i\leq q}$ be the partition of unity subordinate to the cover $\{U_i\}_{1\leq i\leq q}$. Let ω be a Kähler metric on $\overline{M}$ and let C be a positive constant. Then for C large, the Kähler form

$$\omega_p = C\omega + \sum_{i=1}^{p} \sqrt{-1}\partial\overline{\partial}\left(\eta_i \log\log \frac{1}{z_1^i \cdots z_{n_i}^i}\right)$$

defines a complete metric on M with finite volume since on each U_i with $1 \leq i \leq p$, ω_p is bounded from above and below by the local Poincaré metric on U_i. We call this metric the asymptotic Poincaré metric.

The signs of the curvatures of the above metrics are all unknown. We actually only know that the Kähler-Einstein metric has constant negative Ricci curvature and that the McMullen metric has bounded geometry. Also except the asymptotic Poincaré metric, the boundary behaviors of the other metrics are unknown either before our works. It is interesting that to understand them we need to introduce new metrics.

Now we define the Ricci metric and the perturbed Ricci metric. The curvature properties and asymptotics of these two new metrics are understood by us and will be stated in the following sections. Please also see [10] and [11] for details.

By the works of Ahlfors, Royden and Wolpert we know that the Ricci curvature of the Weil-Petersson metric has a negative upper bound. Thus we can use the negative Ricci form of the Weil-Petersson metric as the Kähler form of a new metric. We call this metric the Ricci metric and denote it by τ. That is

$$\omega_\tau = \partial\overline{\partial}\log\omega_{WP}^n.$$

Through careful analysis, we now understand that the Ricci metric is a natural canonical complete Kähler metric with many good properties. However, its holomorphic sectional curvature is only asymptotically negative. To get a metric with good sign on its curvatures, we introduced the perturbed Ricci metric $\omega_{\widetilde{\tau}}$ as a combination of the Ricci metric and the Weil-Petersson metric:

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$$

where C is a large positive constant. As we shall see later that the perturbed Ricci metric has desired curvature properties so that we can put it either on the target or on the domain manifold in Yau's Schwarz lemma, from which we can compare the above metrics.

5. The Curvature Formulas

In this section we describe the harmonic lift of a vector field on the moduli space to the universal curve due to Royden, Siu [16] and Schumacher [15]. Details can also be found in [10]. We then use this method to derive the curvature formula for the Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric.

To compute the curvature of a metric on the moduli space, we need to take derivatives of the metric in the direction of the moduli space. However, it is quite difficult to estimate the curvature by using a formula obtained in such a way. The central idea is to obtain a formula where the derivatives are taken in the fiber direction. We can view the deformation of complex structures on a topological surface as the deformation of the Kähler-Einstein metrics.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. Let $n = 3g-3$ be the complex dimension of $\mathcal{M}_g$. Let $\mathfrak{X}$ be the total space and let $\pi : \mathfrak{X} \to \mathcal{M}_g$ be the projection map.

Let $s_1, \cdots, s_n$ be holomorphic local coordinates near a regular point $s \in \mathcal{M}_g$ and assume that z is a holomorphic local coordinate on the fiber $X_s = \pi^{-1}(s)$. For holomorphic vector fields $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$, there are vector fields $v_1, \cdots, v_n$ on $\mathfrak{X}$ such that

(1) $\pi_*(v_i) = \frac{\partial}{\partial s_i}$ for $i = 1, \cdots, n$;

(2) $\overline{\partial} v_i$ are harmonic TX_s-valued $(0,1)$ forms for $i = 1, \cdots, n$.

The vector fields $v_1, \cdots, v_n$ are called the harmonic lift of the vectors $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$. The existence of such harmonic vector fields was pointed by Siu. Schumacher in his work gave an explicit construction of such lift. We now describe it.

Since $g \geq 2$, we can assume that each fiber is equipped with the Kähler-Einstein metric $\lambda = \frac{\sqrt{-1}}{2}\lambda(z,s)dz \wedge d\overline{z}$. The Kähler-Einstein condition gives the following equation:

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda. \tag{5.1}$$

For the rest of this paper we denote $\frac{\partial}{\partial s_i}$ by ∂_i and $\frac{\partial}{\partial z}$ by ∂_z. Let

$$a_i = -\lambda^{-1}\partial_i \partial_{\overline{z}} \log \lambda$$

and let

$$A_i = \partial_{\overline{z}} a_i.$$

Then the harmonic horizontal lift of ∂_i is

$$v_i = \partial_i + a_i \partial_z.$$

In particular

$$B_i = A_i \partial_z \otimes d\overline{z} \in H^1(X_s, T_{X_s})$$

is harmonic. Furthermore, the lift $\partial_i \mapsto B_i$ gives the Kodaira-Spencer map $T_s\mathcal{M}_g \to H^1(X_s, T_{X_s})$. Thus the Weil-Petersson metric on $\mathcal{M}_g$ is

$$h_{i\overline{j}}(s) = \int_{X_s} B_i \cdot \overline{B_j}\ dv = \int_{X_s} A_i \overline{A_j}\ dv,$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form on the fiber X_s.

Let $R_{i\overline{j}k\overline{l}}$ be the curvature tensor of the Weil-Petersson metric. Here we adopt the following notation for the curvature of a Kähler metric:

For a Kähler metric (M, g), the curvature tensor is given by

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} - g^{p\overline{q}} \frac{\partial g_{i\overline{q}}}{\partial z_k} \frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}.$$

In this case, the Ricci curvature is given by

$$R_{i\overline{j}} = -g^{k\overline{l}} R_{i\overline{j}k\overline{l}}.$$

By using the curvature of the Weil-Petersson metric, we can define the Ricci metric:

$$\tau_{i\overline{j}} = h^{k\overline{l}} R_{i\overline{j}k\overline{l}}$$

and the perturbed Ricci metric:

$$\widetilde{\tau}_{i\overline{j}} = \tau_{i\overline{j}} + C h_{i\overline{j}}$$

where C is a positive constant.

Before we present the curvature formulas for the above metrics, we need to introduce the Maass operators and norms on a Riemann surface [21].

Let X be a Riemann surface and let κ be its canonical bundle. For any integer p, let $S(p)$ be the space of smooth sections of $(\kappa \otimes \overline{\kappa}^{-1})^{\frac{p}{2}}$. Fix a conformal metric $ds^2 = \rho^2(z)|dz|^2$. In the following, we will take ds^2 to be the Kähler-Einstein metric although the following definitions work for all metrics.

The Maass operators K_p and L_p are defined to be the metric derivatives
$K_p : S(p) \to S(p+1)$ and $L_p : S(p) \to S(p-1)$ given by

$$K_p(\sigma) = \rho^{p-1}\partial_z(\rho^{-p}\sigma)$$

and

$$L_p(\sigma) = \rho^{-p-1}\partial_{\overline{z}}(\rho^p \sigma)$$

where $\sigma \in S(p)$.

The operators $P = K_1 K_0$ and $\Box = -L_1 K_0$ will play important roles in the curvature formulas. Here the operator $\Box$ is just the Laplace operator. We also let $T = (\Box + 1)^{-1}$ to be the Green operator.

Each element $\sigma \in S(p)$ have a well-defined absolute value $|\sigma|$ which is independent of the choice of local coordinate. We define the C^k norm of σ:
Let Q be an operator which is a composition of operators K_* and L_*. Denote by $|Q|$ the number of factors. For any $\sigma \in S(p)$, define

$$\|\sigma\|_0 = \sup_X |\sigma|$$

and

$$\|\sigma\|_k = \sum_{|Q| \le k} \|Q\sigma\|_0.$$

We can also localize the norm on a subset of X. Let $\Omega \subset X$ be a domain. We can define

$$\|\sigma\|_{0,\Omega} = \sup_\Omega |\sigma|$$

and

$$\|\sigma\|_{k,\Omega} = \sum_{|Q| \le k} \|Q\sigma\|_{0,\Omega}.$$

We let $f_{i\bar{j}} = A_i \overline{A}_j$ and $e_{i\bar{j}} = T(f_{i\bar{j}})$. These functions will be the building blocks for the curvature formulas.

The trick of converting derivatives from the moduli directions to the fiber directions is the following lemma due to Siu and Schumacher:

Lemma 5.1. *Let η be a relative $(1,1)$-form on the total space $\mathfrak{X}$. Then*

$$\frac{\partial}{\partial s_i} \int_{X_s} \eta = \int_{X_s} L_{v_i} \eta.$$

The curvature formula of the Weil-Petersson metric was first established by Wolpert by using a different method [19] and later was generalized by Siu [16] and Schumacher [15] by using the above lemma:

Theorem 5.1. *The curvature of the Weil-Petersson metric is given by*

$$R_{i\bar{j}k\bar{l}} = \int_{X_s} \left(e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}} \right) dv. \tag{5.2}$$

For the proof, please see [10]. From this formula it is rather easy to show that the Ricci and the holomorphic sectional curvature have explicit negative upper bound.

To establish the curvature formula of the Ricci metric, we need to introduce more operators. Firstly, the commutator of the operator v_k and $(\Box + 1)$ will play an important role. Here we view the vector field v_k as a operator acting on functions. We define

$$\xi_k = [\Box + 1, v_k].$$

A direct computation shows that

$$\xi_k = -A_k P.$$

Also we can define the commutator of $\overline{v_l}$ and ξ_k. Let

$$Q_{k\bar{l}} = [\overline{v_l}, \xi_k].$$

We have

$$Q_{k\bar{l}}(f) = \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z f_{k\bar{l}} \partial_{\bar{z}} f$$

for any smooth function f.

To simplify the notation, we introduce the symmetrization operator of the indices. Let U be any quantity which depends on indices $i,k,\alpha,\overline{j},\overline{l},\overline{\beta}$. The symmetrization operator σ_1 is defined by taking summation of all orders of the triple (i,k,α). That is

$$\begin{aligned}\sigma_1(U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta})) =&U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta})+U(i,\alpha,k,\overline{j},\overline{l},\overline{\beta})+U(k,i,\alpha,\overline{j},\overline{l},\overline{\beta})+U(k,\alpha,i,\overline{j},\overline{l},\overline{\beta})\\&+U(\alpha,i,k,\overline{j},\overline{l},\overline{\beta})+U(\alpha,k,i,\overline{j},\overline{l},\overline{\beta}).\end{aligned}$$

Similarly, σ_2 is the symmetrization operator of $\overline{j}$ and $\overline{\beta}$ and $\widetilde{\sigma_1}$ is the symmetrization operator of $\overline{j}$, $\overline{l}$ and $\overline{\beta}$.

In [10] the following curvature formulas for the Ricci and perturbed Ricci metric were proved:

Theorem 5.2. *Let $s_1,\cdots,s_n$ be local holomorphic coordinates at $s\in M_g$. Then at s, we have*

$$\begin{aligned}\widetilde{R}_{i\bar{j}k\bar{l}} =&h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{(\square+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+(\square+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\right\}\,dv\right\}\\&+h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\right\}\\&-\tau^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\right\}\\&+\tau_{p\bar{j}}h^{p\overline{q}}R_{i\overline{q}k\bar{l}}\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}P_{i\bar{j}k\bar{l}} =&h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{(\square+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+(\square+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\right\}\,dv\right\}\\&+h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\right\}\\&-\widetilde{\tau}^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\right\}\\&+\tau_{p\bar{j}}h^{p\overline{q}}R_{i\overline{q}k\bar{l}}+CR_{i\bar{j}k\bar{l}}.\end{aligned}\tag{5.4}$$

where $R_{i\bar{j}k\bar{l}}$, $\widetilde{R}_{i\bar{j}k\bar{l}}$, and $P_{i\bar{j}k\bar{l}}$ are the curvature of the Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric respectively.

Unlike the curvature formula of the Weil-Petersson metric which we can see the sign of the curvature directly, the above formulas are too complicated and we cannot see the sign. So we need to study the asymptotic behaviors of these curvatures, and first the metrics themselves.

6. The Asymptotics

To compute the asymptotics of these metrics and their curvatures, we first need to find a canonical way to construct local coordinates near the boundary of the moduli space. We first describe the Deligne-Mumford compactification of the moduli space and introduce the pinching coordinate and the plumbing construction which due to Earle and Marden. Please see [12], [21], [17] and [10] for details.

A point p in a Riemann surface X is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v)\mid uv=0,\ |u|,|v|<1\}\subset\mathbb{C}^2$. Let $p_1,\cdots,p_k$ be the nodes on X. X is called stable if each connected component of $X\setminus\{p_1,\cdots,p_k\}$ has negative Euler characteristic. Namely, each connected component has a unique complete hyperbolic metric.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g\geq 2$. The Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ is the union of $\mathcal{M}_g$ and corresponding stable nodal surfaces [3]. Each point $y\in\overline{\mathcal{M}}_g\setminus\mathcal{M}_g$ corresponds to a stable noded surface X_y.

We recall the rs-coordinate on a Riemann surface defined by Wolpert in [21]. There are two cases: the puncture case and the short geodesic case. For the puncture case, we have a noded surface X and a node $p \in X$. Let a, b be two punctures which are paired to form p.

Definition 6.1. *The local coordinate charts (U, u) near a is called rs-coordinate if $u(a) = 0$, u maps U to the punctured disc $0 < |u| < c$ with $c > 0$ and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.*

For the short geodesic case, we have a closed surface X, a closed geodesic $\gamma \subset X$ with length $l < c_*$ where c_* is the collar constant.

Definition 6.2. *The local coordinate chart (U, z) is called rs-coordinate at γ if $\gamma \subset U$, z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$ and the Kähler-Einstein metric on X can be written as $\frac{1}{2}(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|})^2|dz|^2$.*

Remark 6.1. We put the factor $\frac{1}{2}$ in the above two definitions to normalize such that (5.1) holds.

By Keen's collar theorem [5], we have the following lemma:

Lemma 6.1. *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

(1) *z maps Ω to the annulus $\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c$ for $c > 0$;*
(2) *the Kähler-Einstein metric on X restrict to Ω is given by*

$$(\frac{1}{2}u^2 r^{-2}\csc^2\tau)|dz|^2 \tag{6.1}$$

where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;
(3) *the geodesic γ is given by $|z| = e^{-\frac{\pi^2}{l}}$.*

We call such a collar Ω a genuine collar.

We notice that the constant c in the above lemma has a lower bound such that the area of Ω is bounded from below. Also, the coordinate z in the above lemma is rs-coordinate. In the following, we will keep the notation u, r and τ.

Now we describe the local manifold cover of $\overline{\mathcal{M}}_g$ near the boundary. We take the construction of Wolpert [21]. Let $X_{0,0}$ be a noded surface corresponding to a codimension m boundary point. $X_{0,0}$ have m nodes $p_1, \cdots, p_m$. $X_0 = X_{0,0} \setminus \{p_1, \cdots, p_m\}$ is a union of punctured Riemann surfaces. Fix rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \cdots, m$ such that all the U_i and V_i are mutually disjoint. Now pick an open set $U_0 \subset X_0$ such that the intersection of each connected component of X_0 and U_0 is a nonempty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. Now pick Beltrami differentials $\nu_{m+1}, \cdots, \nu_n$ which are supported in U_0 and span the tangent space at X_0 of the deformation space of X_0. For $s = (s_{m+1}, \cdots, s_n)$, let $\nu(s) = \sum_{i=m+1}^n s_i\nu_i$. We assume $|s| = (\sum|s_i|^2)^{\frac{1}{2}}$ is small enough such that $|\nu(s)| < 1$. The noded surface $X_{0,s}$ is obtained by solving the Beltrami equation $\overline{\partial}w = \nu(s)\partial w$. Since $\nu(s)$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates on $X_{0,s}$. Note that they are no longer rs-coordinates. By the theory of Alhfors and Bers [1] and Wolpert [21] we can assume that there are constants $\delta, c > 0$ such that when $|s| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,s}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$. Now we assume $t = (t_1, \cdots, t_m)$ has small norm. We do the plumbing construction on $X_{0,s}$ to obtain $X_{t,s}$. Remove from $X_{0,s}$ the discs $0 < |\eta_i| \le \frac{|t_i|}{c}$ and $0 < |\zeta_i| \le \frac{|t_i|}{c}$ for each $i = 1, \cdots, m$. Now identify $\frac{|t_i|}{c} < |\eta_i| < c$ with $\frac{|t_i|}{c} < |\zeta_i| < c$ by the rule $\eta_i\zeta_i = t_i$. This defines the surface $X_{t,s}$. The tuple $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar defined by $\frac{|t_i|}{c} < |\eta_i| < c$ the plumbing collar.

Remark 6.2. By the estimate of Wolpert [20], [21] on the length of short geodesic, the quantity $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log|t_i|}$.

Now we describe the estimates of the asymptotics of these metrics and their curvatures. The principle is that, when we work on a nearly degenerated surface, the geometry focuses on the collars. Our curvature formulas depend on the Kähler-Einstein metrics of the family of Riemann surfaces near a boundary points. One can obtain approximate Kähler-Einstein metric on these collars by the graft construction of Wolpert [21] which is done by gluing the hyperbolic metric on the nodal surface with the model metric described above.

To use the curvature formulas (5.2), (5.3) and (5.4) to estimate the asymptotic behavior, one also needs to analyze the transition from the plumbing coordinates on the collars to the rs-coordinates. The harmonic Beltrami differentials were constructed by Masur [12] by using the plumbing coordinates and it is easier to compute the integration by using rs-coordinates. Such computation was done in [17] by using the graft metric of Wolpert and the maximum principle. A clear description can be found in [10]. We have the following theorem:

Theorem 6.1. *Let (t,s) be the pinching coordinates on $\overline{\mathcal{M}}_g$ near $X_{0,0}$ which corresponds to a codimension m boundary point of $\overline{\mathcal{M}}_g$. Then there exist constants $M,\delta > 0$ and $1 > c > 0$ such that if $|(t,s)| < \delta$, then the j-th plumbing collar on $X_{t,s}$ contains the genuine collar Ω_c^j. Furthermore, one can choose rs-coordinate z_j on the collar Ω_c^j properly such that the holomorphic quadratic differentials $\psi_1,\cdots,\psi_n$ corresponding to the cotangent vectors $dt_1,\cdots,ds_n$ have form $\psi_i = \varphi_i(z_j)dz_j^2$ on the genuine collar Ω_c^j for $1 \le j \le m$ where*

(1) $\varphi_i(z_j) = \frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $i \ge m+1$;
(2) $\varphi_i(z_j) = (-\frac{t_j}{\pi})\frac{1}{z_j^2}(q_j(z_j) + \beta_j)$ *if* $i = j$;
(3) $\varphi_i(z_j) = (-\frac{t_i}{\pi})\frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $1 \le i \le m$ *and* $i \ne j$.

Here β_i^j and β_j are functions of (t,s), q_i^j and q_j are functions of (t,s,z_j) given by

$$q_i^j(z_j) = \sum_{k<0} \alpha_{ik}^j(t,s) t_j^{-k} z_j^k + \sum_{k>0} \alpha_{ik}^j(t,s) z_j^k$$

and

$$q_j(z_j) = \sum_{k<0} \alpha_{jk}(t,s) t_j^{-k} z_j^k + \sum_{k>0} \alpha_{jk}(t,s) z_j^k$$

such that

(1) $\sum_{k<0} |\alpha_{ik}^j| c^{-k} \le M$ *and* $\sum_{k>0} |\alpha_{ik}^j| c^k \le M$ *if* $i \ne j$;
(2) $\sum_{k<0} |\alpha_{jk}| c^{-k} \le M$ *and* $\sum_{k>0} |\alpha_{jk}| c^k \le M$;
(3) $|\beta_i^j| = O(|t_j|^{\frac{1}{2}-\epsilon})$ *with* $\epsilon < \frac{1}{2}$ *if* $i \ne j$;
(4) $|\beta_j| = (1 + O(u_0))$

where $u_0 = \sum_{i=1}^m u_i + \sum_{j=m+1}^n |s_j|$.

By definition, the metric on the cotangent bundle induced by the Weil-Petersson metric is given by

$$h^{i\bar{j}} = \int_{X_{t,s}} \lambda^{-2} \varphi_i \overline{\varphi}_j \, dv.$$

We then have the following series of estimates, see [10]. First by using this formula and taking inverse, we can estimate the Weil-Petersson metric.

Theorem 6.2. *Let (t,s) be the pinching coordinates. Then*

(1) $h^{i\bar{i}} = 2u_i^{-3}|t_i|^2(1+O(u_0))$ *and* $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1+O(u_0))$ *for* $1 \le i \le m$;

(2) $h^{i\bar{j}} = O(|t_it_j|)$ *and* $h_{i\bar{j}} = O(\frac{u_i^3u_j^3}{|t_it_j|})$ *if* $1 \le i,j \le m$ *and* $i \ne j$;

(3) $h^{i\bar{j}} = O(1)$ *and* $h_{i\bar{j}} = O(1)$ *if* $m+1 \le i,j \le n$;

(4) $h^{i\bar{j}} = O(|t_i|)$ *and* $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \le m < j$ *or* $j \le m < i$.

Then we use the duality to construct the harmonic Beltrami differentials. We have

Lemma 6.2. *On the genuine collar Ω_c^j for c small, the coefficient functions A_i of the harmonic Beltrami differentials have the form:*

(1) $A_i = \frac{z_j}{\bar{z}_j}\sin^2\tau_j\big(\overline{p_i^j(z_j)} + \overline{b_i^j}\big)$ *if* $i \ne j$;

(2) $A_j = \frac{z_j}{\bar{z}_j}\sin^2\tau_j\big(\overline{p_j(z_j)} + \overline{b_j}\big)$

where

(1) $p_i^j(z_j) = \sum_{k\le -1} a_{ik}^j\rho_j^{-k}z_j^k + \sum_{k\ge 1} a_{ik}^j z_j^k$ *if* $i \ne j$;

(2) $p_j(z_j) = \sum_{k\le -1} a_{jk}\rho_j^{-k}z_j^k + \sum_{k\ge 1} a_{jk}z_j^k$.

In the above expressions, $\rho_j = e^{-\frac{2\pi^2}{l_j}}$ and the coefficients satisfy the following conditions:

(1) $\sum_{k\le -1}|a_{ik}^j|c^{-k} = O(u_j^{-2})$ *and* $\sum_{k\ge 1}|a_{ik}^j|c^k = O(u_j^{-2})$ *if* $i \ge m+1$;

(2) $\sum_{k\le -1}|a_{ik}^j|c^{-k} = O(u_j^{-2})O\big(\frac{u_i^3}{|t_i|}\big)$ *and* $\sum_{k\ge 1}|a_{ik}^j|c^k = O(u_j^{-2})O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \le m$ *and* $i \ne j$;

(3) $\sum_{k\le -1}|a_{jk}|c^{-k} = O\big(\frac{u_j}{|t_j|}\big)$ *and* $\sum_{k\ge 1}|a_{jk}|c^k = O\big(\frac{u_j}{|t_j|}\big)$;

(4) $|b_i^j| = O(u_j)$ *if* $i \ge m+1$;

(5) $|b_i^j| = O(u_j)O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \le m$ *and* $i \ne j$;

(6) $b_j = -\frac{u_j}{\pi \bar{t}_j}(1+O(u_0))$.

To use the curvature formulas to estimate the Ricci metric and the perturbed Ricci metric, one needs to find accurate estimate of the operator $T = (\Box + 1)^{-1}$. More precisely, one needs to estimate the functions $e_{i\bar{j}} = T(f_{i\bar{j}})$. To avoid writing down the Green function of T, we construct approximate solutions and localize on the collars in [10]. Pick a positive constant $c_1 < c$ and define the cut-off function $\eta \in C^\infty(\mathbb{R},[0,1])$ by

$$\begin{cases} \eta(x) = 1 & x \le \log c_1 \\ \eta(x) = 0 & x \ge \log c \\ 0 < \eta(x) < 1 & \log c_1 < x < \log c. \end{cases} \tag{6.2}$$

It is clear that the derivatives of η are bounded by constants which only depend on c and c_1. Let $\widetilde{e_{i\bar{j}}}(z)$ be the function on X defined in the following way where z is taken to be z_i on the collar Ω_c^i:

(1) if $i \le m$ and $j \ge m+1$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i\overline{b_i}b_j^i & z \in \Omega_{c_1}^i \\ (\frac{1}{2}\sin^2\tau_i\overline{b_i}b_j^i)\eta(\log r_i) & z \in \Omega_c^i \text{ and } c_1 < r_i < c \\ (\frac{1}{2}\sin^2\tau_i\overline{b_i}b_j^i)\eta(\log\rho_i - \log r_i) & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i \\ 0 & z \in X \setminus \Omega_c^i \end{cases}$$

(2) if $i, j \leq m$ and $i \neq j$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i & z \in \Omega_{c_1}^i \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log r_i) & z \in \Omega_c^i \text{ and } c_1 < r_i < c \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log\rho_i - \log r_i) & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i \\ \frac{1}{2}\sin^2\tau_j \overline{b_i^j} b_j & z \in \Omega_{c_1}^j \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i^j} b_j)\eta(\log r_j) & z \in \Omega_c^j \text{ and } c_1 < r_j < c \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i^j} b_j)\eta(\log\rho_j - \log r_j) & z \in \Omega_c^j \text{ and } c^{-1}\rho_j < r_j < c_1^{-1}\rho_j \\ 0 & z \in X \setminus (\Omega_c^i \cup \Omega_c^j) \end{cases}$$

(3) if $i \leq m$, then

$$\widetilde{e_{i\bar{i}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i |b_i|^2 & z \in \Omega_{c_1}^i \\ (\frac{1}{2}\sin^2\tau_i |b_i|^2)\eta(\log r_i) & z \in \Omega_c^i \text{ and } c_1 < r_i < c \\ (\frac{1}{2}\sin^2\tau_i |b_i|^2)\eta(\log\rho_i - \log r_i) & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i \\ 0 & z \in X \setminus \Omega_c^i \end{cases}$$

Also, let $\widetilde{f_{i\bar{j}}} = (\Box + 1)\widetilde{e_{i\bar{j}}}$. It is clear that the supports of these approximation functions are contained in the corresponding collars. We have the following estimates:

Lemma 6.3. *Let $\widetilde{e_{i\bar{j}}}$ be the functions constructed above. Then*

(1) $e_{i\bar{i}} = \widetilde{e_{i\bar{i}}} + O\big(\frac{u_i^4}{|t_i|^2}\big)$ *if $i \leq m$;*

(2) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$ *if $i, j \leq m$ and $i \neq j$;*

(3) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3}{|t_i|}\big)$ *if $i \leq m$ and $j \geq m+1$;*

(4) $\|e_{i\bar{j}}\|_0 = O(1)$ *if $i, j \geq m+1$.*

Now we use the approximation functions $\widetilde{e}_{i\bar{j}}$ in the formulas (5.2), (5.3) and (5.4). The following theorems were proved in [10] and [11]. We first have the asymptotic estimate of the Ricci metric:

Theorem 6.3. *Let (t, s) be the pinching coordinates. Then we have*

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1 + O(u_0))$ *and* $\tau^{i\bar{i}} = \frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1 + O(u_0))$ *if $i \leq m$;*

(2) $\tau_{i\bar{j}} = O\bigg(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j)\bigg)$ *and* $\tau^{i\bar{j}} = O(|t_i t_j|)$ *if $i, j \leq m$ and $i \neq j$;*

(3) $\tau_{i\bar{j}} = O\big(\frac{u_i^2}{|t_i|}\big)$ *and* $\tau^{i\bar{j}} = O(|t_i|)$ *if $i \leq m$ and $j \geq m+1$;*

(4) $\tau_{i\bar{j}} = O(1)$ *if $i, j \geq m+1$.*

By the asymptotics of the Ricci metric in the above theorem, we have

Corollary 6.1. *There is a constant $C > 0$ such that*

$$C^{-1}\omega_P \leq \omega_\tau \leq \omega_P.$$

Next we estimate the holomorphic sectional curvature of the Ricci metric:

Theorem 6.4. *Let $X_0 \in \overline{\mathcal{M}_g}\backslash\mathcal{M}_g$ be a codimension m point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates at X_0 where $t_1, \cdots, t_m$ correspond to the degeneration directions. Then the holomorphic sectional curvature is negative in the degeneration directions and is bounded in the non-degeneration directions. Precisely, there is a $\delta > 0$ such that if $|(t, s)| < \delta$, then*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0)) > 0 \tag{6.3}$$

if $i \le m$ and

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \tag{6.4}$$

if $i \ge m+1$.

Furthermore, on $\mathcal{M}_g$, the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.

This theorem was proved in [10] by using the formula (5.3) and estimating error terms. However, the holomorphic sectional curvature of the Ricci metric is not always negative. We need to introduce and study the perturbed Ricci metric. We have

Theorem 6.5. *For suitable choice of positive constant C, the perturbed Ricci metric $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ is complete and comparable with the asymptotic Poincaré metric. Its bisectional curvature is bounded. Furthermore, its holomorphic sectional curvature and Ricci curvature are bounded from above and below by negative constants.*

Remark 6.3. The perturbed Ricci metric is the first complete Kähler metric on the moduli space with bounded curvature and negatively pinched holomorphic sectional curvature and Ricci curvature.

By using the minimal surface theory and Bers' embedding theorem, we have also proved the following theorem in [11]:

Theorem 6.6. *The moduli space equipped with either the Ricci metric or the perturbed Ricci metric has finite volume. The Teichmüller space equipped with either of these metrics has bounded geometry.*

7. The Equivalence of the Complete Metrics

In this section we describe our arguments that all of the complete metrics on the Teichmüller space and moduli space discussed above are equivalent. With the good understanding of the Ricci and the perturbed Ricci metrics, the results of this section are quite easy consequences of Yau's Schwarz lemma and also the basic definitions of these metrics. We first give the definition of equivalence of metrics:

Definition 7.1. *Two Kähler metrics g_1 and g_2 on a manifold X are equivalent or two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the tangent bundle of X are equivalent if there is a constant $C > 0$ such that*

$$C^{-1} g_1 \le g_2 \le C g_1$$

or

$$C^{-1}\|\cdot\|_1 \le \|\cdot\|_2 \le C\|\cdot\|_1.$$

We denote this by $g_1 \sim g_2$ or $\|\cdot\|_1 \sim \|\cdot\|_2$.

The main result of this section we want to discuss is the following theorem proved in [10] and [11]:

Theorem 7.1. *On the moduli space $\mathcal{M}_g$ ($g \ge 2$), the Teichmüller metric $\|\cdot\|_T$, the Carathéodory metric $\|\cdot\|_C$, the Kobayashi metric $\|\cdot\|_K$, the Kähler-Einstein metric ω_{KE}, the induced Bergman metric ω_B, the McMullen metric ω_M, the asymptotic Poincaré metric ω_P the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\widetilde{\tau}}$ are equivalent. Namely*

$$\omega_{KE} \sim \omega_{\widetilde{\tau}} \sim \omega_\tau \sim \omega_P \sim \omega_B \sim \omega_M$$

and

$$\|\cdot\|_K = \|\cdot\|_T \sim \|\cdot\|_C \sim \|\cdot\|_M.$$

As corollary we proved the following conjecture of Yau made in the early 80s [24], [14]:

Theorem 7.2. *The Kähler-Einstein metric is equivalent to the Teichmüller metric on the moduli space:* $\|\cdot\|_{KE} \sim \|\cdot\|_T$.

Another corollary was also conjectured by Yau as one of his 120 famous problems [24], [14]:

Theorem 7.3. *The Kähler-Einstein metric is equivalent to the Bergman metric on the Teichmüller space:* $\omega_{KE} \sim \omega_B$.

Now we briefly describe the idea of proving the comparison theorem. To compare two complete metrics on a noncompact manifold, we need to write down their asymptotic behavior and compare near infinity. However, if one can not find the asymptotics of these metrics, the only tool we have is the following Yau's Schwarz lemma [22]:

Theorem 7.4. *Let* $f : (M^m, g) \to (N^n, h)$ *be a holomorphic map between Kähler manifolds where* M *is complete and* $Ric(g) \geq -c\, g$ *with* $c \geq 0$.

(1) *If the holomorphic sectional curvature of* N *is bounded above by a negative constant, then* $f^*h \leq \tilde{c}\, g$ *for some constant* $\tilde{c}$.
(2) *If* $m = n$ *and the Ricci curvature of* N *is bounded above by a negative constant, then* $f^*\omega_h^n \leq \tilde{c}\, \omega_g^n$ *for some constant* $\tilde{c}$.

We briefly describe the proof of the comparison theorem by using Yau's Schwarz lemma and the curvature computations and estimates.

Sketch of proof. To use this result, we take $M = N = \mathcal{M}_g$ and let f be the identity map. We know the perturbed Ricci metric is obtained by adding a positive Kähler metric to the Ricci metric. Thus it is bounded from below by the Ricci metric.

Consider the identity map

$$id : (\mathcal{M}_g, \omega_\tau) \to (\mathcal{M}_g, \omega_{WP}).$$

Yau's Schwarz Lemma implies $\omega_{WP} \leq C_0 \omega_\tau$. So

$$\omega_\tau \leq \omega_{\tilde{\tau}} = \omega_\tau + C\omega_{WP} \leq (CC_0 + 1)\omega_\tau.$$

Thus $\omega_\tau \sim \omega_{\tilde{\tau}}$.

To control the Kähler-Einstein metric, we consider

$$id : (\mathcal{M}_g, \omega_{KE}) \to (\mathcal{M}_g, \omega_{\tilde{\tau}})$$

and

$$id : (\mathcal{M}_g, \omega_{\tilde{\tau}}) \to (\mathcal{M}_g, \omega_{KE}).$$

Yau's Schwarz Lemma implies

$$\omega_{\tilde{\tau}} \leq C_0 \omega_{KE}$$

and

$$\omega_{KE}^n \leq C_0 \omega_{\tilde{\tau}}^n.$$

The equivalence follows from linear algebra.

Thus by Corollary 6.1 we have

$$\omega_{KE} \sim \omega_{\tilde{\tau}} \sim \omega_\tau \sim \omega_P.$$

By using similar method we have $\omega_\tau \leq C\omega_M$. To show the other side of the inequality, we have to analyze the asymptotic behavior of the geodesic length functions. We showed in [10] that

$$\omega_\tau \sim \omega_M.$$

Thus by the work of McMullen [13] we have

$$\omega_\tau \sim \omega_M \sim \|\cdot\|_T.$$

The work of Royden showed that the Teichmüller metric coincides with the Kobayashi metric. Thus we need to show that the Carathéodory metric and the Bergman metric are comparable with the Kobayashi metric. This was done in [11] by using Bers' embedding theorem. The idea is as follows:

By the Bers' embedding theorem, for each point $p \in \mathcal{T}_g$, there is a map $f_p : \mathcal{T}_g \to \mathbb{C}^n$ such that $f_p(p) = 0$ and

$$B_2 \subset f_p(\mathcal{T}_g) \subset B_6$$

where B_r is the open ball in $\mathbb{C}^n$ centered at 0 with radius r. Since both Carathéodory metric and Kobayashi metric have restriction property and can be computed explicitly on balls, we can use these metrics defined on B_2 and B_6 to pinch these metrics on the Teichmüller space. We can also use this method to estimate peak sections of the Teichmüller space at the point p. A careful analysis shows

$$\|\cdot\|_C \sim \|\cdot\|_K \sim \omega_B.$$

The argument is quite easy. Please see [11] for details.

□

8. Bounded Geometry of the Kähler-Einstein Metric

The comparison theorem gives us some control on the Kähler-Einstein Metric. Especially we know that it has Poincaré growth near the boundary of the moduli space and is equivalent to the Ricci metric which has bounded geometry. In this section we sketch our proof that the Kähler-Einstein metric also has bounded geometry. Precisely we have

Theorem 8.1. *The curvature of the Kähler-Einstein metric and all of its covariant derivatives are uniformly bounded on the Teichmüller spaces, and its injectivity radius has lower bound.*

Now we briefly describe the proof. Please see [11] for details.

Sketch of proof. We follow Yau's argument in [23]. The first step is to perturb the Ricci metric using Kähler-Ricci flow

$$\begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}) \\ g(0) = \tau \end{cases}$$

to avoid complicated computations of the covariant derivatives of the curvature of the Ricci metric.

For $t > 0$ small, let $h = g(t)$ and let g be the Kähler-Einstein metric. We have

(1) h is equivalent to the initial metric τ and thus is equivalent to the Kähler-Einstein metric.
(2) The curvature and its covariant derivatives of h are bounded.

Then we consider the Monge-Amperé equation

$$\log\det(h_{i\bar{j}} + u_{i\bar{j}}) - \log\det(h_{i\bar{j}}) = u + F$$

where $\partial\overline{\partial}u = \omega_g - \omega_h$ and $\partial\overline{\partial}F = Ric(h) + \omega_h$.

The curvature of $P_{i\bar{j}k\bar{l}}$ of the Kähler-Einstein metric is given by

$$P_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + u_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}} + u_{;i\bar{j}k\bar{l}} - g^{p\bar{q}}u_{;i\bar{q}k}u_{;\bar{j}p\bar{l}}.$$

The comparison theorem implies $\partial\overline{\partial}u$ has C^0-bound and the strong bounded geometry of h implies $\partial\overline{\partial}F$ has C^k-bound for $k \geq 0$. Also, the equivalence of h and g implies $u+F$ is bounded.

So we need the C^k-bound of $\partial\overline{\partial}u$ for $k \geq 1$. Let

$$S = g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}u_{;i\bar{q}k}u_{;\bar{j}p\bar{l}}$$

and

$$V = g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}\left(u_{;i\bar{q}k\bar{n}}u_{;\bar{j}p\bar{l}m} + u_{;i\bar{n}kp}u_{;\bar{j}m\bar{l}\bar{q}}\right)$$

where the covariant derivatives of u were taken with respect to the metric h.

Yau's C^3 estimate in [23] implies S is bounded. Let $f = (S+\kappa)V$ where κ is a large constant. The inequality

$$\Delta' f \geq Cf^2 + (\text{ lower order terms })$$

implies f is bounded and thus V is bounded. So the curvature of the Kähler-Einstein metric are bounded. Same method can be used to derive boundedness of higher derivatives of the curvature.

□

Actually we have also proved the all of these complete Kähler metrics have bounded geometry, which should be useful in understanding the geometry of the moduli and the Teichmüller spaces.

9. Application to Algebraic Geometry

The existence of the Kähler-Einstein metric is closely related to the stability of the tangent and cotangent bundle. In this section we review our results that the logarithmic extension of the cotangent bundle of the moduli space is stable in the sense of Mumford. We first recall the definition. Please see [6] for details.

Definition 9.1. *Let E be a holomorphic vector bundle over a complex manifold X and let Φ be a Kähler class of X. The (Φ-)degree of E is given by*

$$\deg(E) = \int_X c_1(E)\Phi^{n-1}$$

where n is the dimension of X. The slope of E is given by the quotient

$$\mu(E) = \frac{\deg(E)}{rank(E)}.$$

The bundle E is Mumford (Φ-)stable if for any proper coherent subsheaf $\mathcal{F} \subset E$, we have

$$\mu(\mathcal{F}) < \mu(E).$$

Now we describe the logarithmic cotangent bundle. Let U be any local chart of $\mathcal{M}_g$ near the boundary with pinching coordinates $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ such that $(t_1, \cdots, t_m)$ represent the degeneration directions. Let

$$e_i = \begin{cases} \frac{dt_i}{t_i} & i \leq m; \\ ds_i & i \geq m+1. \end{cases}$$

The logarithmic cotangent bundle E is the extension of $T^*\mathcal{M}_g$ to $\overline{\mathcal{M}}_g$ such that on U, $e_1, \cdots, e_n$ is a local holomorphic frame of E. One can write down the transition functions and check that there is a unique bundle over $\overline{\mathcal{M}}_g$ satisfing the above condition.

To prove the stability of E, we need to specify a Kähler class. It is natural to use the polarization of E. The main theorem of this section is the following:

Theorem 9.1. *The first Chern class $c_1(E)$ is positive and E is stable with respect to $c_1(E)$.*

We briefly describe here the proof of this theorem. Please see [11] for details.

Sketch of the proof. Since we only deal with the first Chern class, we can assume the coherent subsheaf $\mathcal{F}$ is actually a subbundle F.

Since the Kähler-Einstein metric induces a singular metric g^*_{KE} on the logarithmic extension bundle E, our main job is to show that the degree and slope of E and any proper subbundle F defined by the singular metric are finite and are equal to the genuine ones. This depends on our estimates of the Kähler-Einstein metric which are used to show the convergence of the integrals defining the degrees.

More precisely we need to show the following:

(1) As a current, $\omega_{_{KE}}$ is closed and represent the first Chern class of E, that is

$$[\omega_{_{KE}}] = c_1(\overline{E}).$$

(2) The singular metric $g^*_{_{KE}}$ on E induced by the Kähler-Einstein metric defines the degree of E

$$\deg(E) = \int_{\mathcal{M}_g} \omega^n_{_{KE}}.$$

(3) The degree of any proper holomorphic sub-bundle F of E can be defined by using $g^*_{_{KE}} \mid_F$,

$$\deg(F) = \int_{\mathcal{M}_g} -\partial\overline{\partial} \log\det\left(g^*_{_{KE}} \mid_F\right) \wedge \omega^{n-1}_{_{KE}}.$$

These three steps were proved in [11] by using the Poincaré growth property of the Kähler-Einstein metric together with a special cut-off function. This shows that the bundle E is semi-stable.

To get the strict stability, we proceeded by contradiction. If E is not stable, then E, thus $E \mid_{\mathcal{M}_g}$, split holomorphically. This implies a finite smooth cover of the moduli space splits which implies a finite index subgroup of the mapping class group splits as a direct product of two subgroups. This is impossible by a topological fact. Again, the detailed proof can be found in [11].

□

10. Final Remarks

Although significant progresses have been made in understanding the geometry of the Teichmüller and the moduli spaces, there are still many problems remain to be solved, such as the goodness of these complete Kähler metrics, the computation of their L^2-cohomology groups, the convergence of the Ricci flow starting from the Ricci metric to the Kähler-Einstein metric, the representations of the mapping class group on the middle dimensional L^2-cohomology of these metrics, and the index theory associated to these complete Kähler metrics. Also the perturbed Ricci metric is the first complete Kähler metric on the moduli spaces with bounded negative Ricci and holomorphic sectional curvature and bounded geometry, we believe this metric must have more interesting applications. Another question is which of these metrics are actually identical. We hope to report on the progresses of the study of these problems on a later occasion.

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