CANONICAL METRICS ON THE MODULI SPACE OF RIEMANN SURFACES II

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Contents

1. Introduction

In [9] we started the project to understand the canonical metrics on the Teichmüller and the moduli spaces of Riemann surfaces, especially the Kähler-Einstein metric. Our goal is to understand the geometry and topology of the moduli spaces from understanding those classical metrics, as well as to find new complete Kähler metrics with good curvature property. In [9] we studied in detail two new complete Kähler metrics, the Ricci and the perturbed Ricci metric. In particular we proved that the Ricci metric has bounded holomorphic bisectional curvature, and the perturbed Ricci metric, not only has bounded holomorphic bisectional curvature, but also has bounded negative holomorphic sectional curvature, and bounded negative Ricci curvature. By using the perturbed Ricci metric as a bridge we were able to prove the equivalence of several classical complete metrics on the Teichmüller and the moduli spaces, including the Teichmüller metric, the Kobayashi metric, the Cheng-Yau-Mok Kähler-Einstein metric, the McMullen metric, as well as the Ricci and the perturbed Ricci metric. This also solved an old conjecture of Yau about the equivalence of the Kähler-Einstein metric and the Teichmüller metric.

In this paper we continue our study on these metrics and other classical metrics, in particular the Kähler-Einstein metric, and the perturbed Ricci metric. One of the main results is the good understanding of the Kähler-Einstein metric, from which we will derive some corollaries about the geometry of the moduli spaces. We will first prove the equivalence of the Bergman metric and the Carathéodory metric to the Kähler-Einstein metric. This completes our project on comparing all of the known complete metrics on the Teichmüller and moduli spaces. We then prove that the Ricci curvature of the perturbed Ricci metric has negative upper and lower bounds, and it also has bounded geometry. Recall that it also has bounded negative holomorphic sectional curvature. The perturbed Ricci metric is the first known complete Kähler metric on the Teichmüller and the moduli space with such good negative curvature property. We then focus on the Kähler-Einstein metric, study in detail its boundary behaviors and prove that not only it has bounded geometry, but also all of the covariant derivatives of its curvature are uniformly bounded. It is natural to expect interesting applications of the good properties of the

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perturbed Ricci and the Kähler-Einstein metric. For example, as an application of our detailed understanding of these metrics we prove that the logarithmic cotangent bundle of the moduli space is stable in the sense of Mumford.

This paper is organized as follows. In Section 2, by using the Bers' embedding theorem, we prove that both the Carathéodory metric and the Bergman metric on the Teichmüller space are equivalent to the Kobayashi metric which is equivalent to the Ricci metric, perturbed Ricci metric, Kähler-Einstein metric, and the McMullen metric by the work in [9]. The equivalence between the Bergman metric and the Kähler-Einstein metric was first conjectured by Yau as Problem 44 in his 120 open problems in geometry [20], [14].

In Section 3, we will prove that both the Ricci metric and the perturbed Ricci metric have bounded curvature. Especially, with a suitable choice of the perturbation constant, the holomorphic sectional curvature and the Ricci curvature of the perturbed Ricci metric are pinched by negative constants. As a simple corollary, we immediately see that the dual of the logarithmic cotangent bundle has no nontrivial holomorphic section. By using Bers' embedding theorem and minimal surface, we also prove that the Teichmüller space equipped with either of these two metrics has bounded geometry: bounded curvature and lower bound of injectivity radius. McMullen proved that the McMullen metric has bounded geometry.

Having a complete Kähler-Einstein metric puts strong restrictions on the geometric structure of the moduli space. In Section 4 we will study the cohomology classes defined by the Kähler forms and Ricci forms of the Ricci metric and the Kähler-Einstein metric. As a direct corollary, we will see easily that the moduli space is of logarithmic general type. One of the most interesting applications of these study is a proof of the stability of the logarithmic cotangent bundle of the moduli space in the sense of Mumford.

Finally in Section 5, we will study the bounded geometry of the Kähler-Einstein metric. We set up the Monge-Amperé equation from a new metric obtained by deforming the Ricci metric along the Kähler-Ricci flow. The work in [9] provides a C^2 estimate. We follow the work of Yau in [19] and do the C^3 and C^4 estimates. These estimates imply that the curvature of the Kähler-Einstein metric is bounded. The same method proves that all of the covariant derivatives of the curvature are bounded. This may be used to understand the complicated boundary of the Teichmüller space.

Now we give the precise statements of the main results in this paper. We fix an integer $g \geq 2$ and denote by $\mathcal{T} = \mathcal{T}_g$ the Teichmüller space, and $\mathcal{M}_g$ the moduli space of closed Riemann surfaces of genus g. Our first result is the following theorem which will be proved in Section 2:

Theorem 1.1. *The Bergman metric and the Carathéodory metric both are equivalent to the Kobayashi metric, therefore to the Kähler-Einstein metric on the Teichmüller space.*

Recall that we say two metrics are equivalent if they are quasi-isometric to each other. We note that the equivalence between the Bergman metric and the Kähler-Einstein metric was conjectured by Yau in [20], [14]. The proof of the first part of Theorem 1.1 only needs the Bers' embedding and the most basic properties of the Bergman and the Carathéodory metrics.

Our second main result, proved in Section 3, is about the curvature properties of the Ricci and the perturbed Ricci metric. We have two theorems, the first one is about the Ricci metric:

Theorem 1.2. *The holomorphic bisectional curvature, the holomorphic sectional curvature and the Ricci curvature of the Ricci metric τ on the moduli space $\mathcal{M}_g$ are bounded.*

And the second theorem is about the perturbed Ricci metric:

Theorem 1.3. *For any constant $C > 0$, the bisectional curvature of the perturbed Ricci metric $\widetilde{\tau} = \tau + Ch$ is bounded. Furthermore, with a suitable choice of C, the holomorphic sectional curvature and the Ricci curvature of $\widetilde{\tau}$ are bounded from above and below by negative constants.*

Both of the above theorems are proved by a detailed analysis of the boundary behavior of the metrics and their curvature. Together with the following result, which is proved by using minimal surface theory and the Bers' embedding, they imply that Ricci metric and the perturbed Ricci metric both have bounded geometry.

Corollary 1.1. *The injectivity radius of the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric is bounded from below.*

Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of the moduli space of Riemann surfaces. Let $\overline{E}$ denote the logarithmic extension of the cotangent bundle of the moduli space. The next result is an interesting consequence of our detailed understanding of the Kähler-Einstein metric on the moduli spaces. It is proved in Section 4:

Theorem 1.4. *The first Chern class of $\overline{E}$ is positive and $\overline{E}$ is Mumford stable with respect to $c_1(\overline{E})$.*

This theorem also implies that $\overline{\mathcal{M}}_g$ is of logarithmic general type for any $g \geq 2$. We prove this theorem by using the good control of the Kähler-Einstein metric and the Ricci metric we obtained near the boundary of the moduli space.

Our final result, proved in Section 5, is the following theorem about the Kähler-Einstein metric:

Theorem 1.5. *The Kähler-Einstein metric on the Teichmüller space $\mathcal{T}_g$ has bounded geometry. Furthermore the covariant derivatives of its curvature are all uniformly bounded.*

This theorem is proved by using the Kähler-Ricci flow and the method of Yau in his proof of the Calabi conjecture to obtain higher order estimates of the curvature.

We will continue our study on the geometry of the moduli space and Teichmüller space. The topics will include the goodness on the moduli space of these metrics and the L^2-cohomology of these metrics and the Weil-Petersson metric on the Teichmüller space, and the convergence of the Kähler-Ricci flow starting from the Ricci and the perturbed Ricci metric.

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2. The Carathéodory Metric and the Bergman Metric

In this section we prove that the Carathéodory metric and the Bergman metric on the Teichmüller space are equivalent to the Kobayashi metric by using the Bers' embedding theorem. This achieves one of our initial goals on the equivalence of all known complete metrics on the Teichmüller space.

We first describe the idea. By the Bers' embedding theorem, we know that for each point p in the Teichmüller space, we can find an embedding map of the Teichmüller space into $\mathbb{C}^n$ such that p is mapped to the origin and the image of the Teichmüller space contains the ball of radius 2 and is contained inside the ball of radius 6. The Kobayashi metric and the Carathéodory metric of these balls coincide and can be computed directly. Also, both of these metrics have restriction property. Roughly speaking, the metrics on a submanifold are larger than those on the ambient manifold. We use explicit form of these metrics on the balls together with this property to estimate the Kobayashi and the Carathéodory metric on the Teichmüller space and compare them on a smaller ball. On the other hand, the norm defined by the Bergman metric at each point can be estimated by using the quotient of peak sections at this point. We use upper and lower bounds of these peak sections to compare the Bergman metric, the Kobayashi metric and the Euclidean metric on a small ball in the image under the Bers' embedding of the Teichmüller space.

At first, we briefly recall the definitions of the Carathéodory, Bergman and Kobayashi metric on a complex manifold. Please see [6] for details.

Let X be a complex manifold and of dimension n. let Δ_R be the disk in $\mathbb{C}$ with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincaré metric on Δ. Let $p \in X$ be a point and let $v \in T_pX$ be a holomorphic tangent vector. Let $\mathrm{Hol}(X, \Delta_R)$ and $\mathrm{Hol}(\Delta_R, X)$ be the spaces of holomorphic maps from X to Δ_R and from Δ_R to X respectively. The Carathéodory norm of the vector v is defined to be

$$\|v\|_C = \sup_{f \in \mathrm{Hol}(X,\Delta)} \|f_* v\|_{\Delta,\rho}$$

and the Kobayashi norm of v is defined to be

$$\|v\|_K = \inf_{f \in \mathrm{Hol}(\Delta_R, X),\ f(0)=p,\ f'(0)=v} \frac{2}{R}.$$

Now we define the Bergman metric on X. Let K_X be the canonical bundle of X and let W be the space of L^2 holomorphic sections of K_X in the sense that if $\sigma \in W$, then

$$\|\sigma\|_{L^2}^2 = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma} < \infty.$$

The inner product on W is defined to be

$$(\sigma, \rho) = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\rho}$$

for all $\sigma, \rho \in W$. Let $\sigma_1, \sigma_2, \cdots$ be an orthonormal basis of W. The Bergman kernel form is the non-negative (n,n)-form

$$B_X = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} \sigma_j \wedge \overline{\sigma}_j.$$

With a choice of local coordinates $z_i, \cdots, z_n$, we have

$$B_X = BE_X(z, \overline{z})(\sqrt{-1})^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

where $BE_X(z,\overline{z})$ is called the Bergman kernel function. If the Bergman kernel B_X is positive, one can define the Bergman metric

$$B_{i\overline{j}} = \frac{\partial^2 \log BE_X(z, \overline{z})}{\partial z_i \partial \overline{z}_j}.$$

The Bergman metric is well-defined and is nondegenerate if the elements in W separate points and the first jet of X.

We will use the following notations:

Definition 2.1. *Let X be a complex space. For each point $p \in X$ and each holomorphic tangent vector $v \in T_pX$, we denote by $\|v\|_{B,X,p}$, $\|v\|_{C,X,p}$ and $\|v\|_{K,X,p}$ the norms of v measured in the Bergman metric, the Carathéodory metric and the Kobayashi metric of the space X respectively.*

Now we fix an integer $g \geq 2$ and denote by $\mathcal{T} = \mathcal{T}_g$ the Teichmüller space of closed Riemann surface of genus g. Our main theorem of this section is the following:

Theorem 2.1. *Let $\mathcal{T}$ be the Teichmüller space of closed Riemann surfaces of genus g with $g \geq 2$. Then there is a positive constant C only depending on g such that for each point $p \in \mathcal{T}$ and each vector $v \in T_p\mathcal{T}$, we have*

$$C^{-1}\|v\|_{K,\mathcal{T},p} \leq \|v\|_{B,\mathcal{T},p} \leq C\|v\|_{K,\mathcal{T},p}$$

and

$$C^{-1}\|v\|_{K,\mathcal{T},p} \leq \|v\|_{C,\mathcal{T},p} \leq C\|v\|_{K,\mathcal{T},p}.$$

Proof. We will show that the norms defined by these metrics are uniformly equivalent at each point of $\mathcal{T}$. We first collect some known results in the following lemma.

Lemma 2.1. *Let X be a complex space. Then*

(1) $\|\cdot\|_{C,X} \leq \|\cdot\|_{K,X}$;
(2) *Let Y be another complex space and $f: X \to Y$ be a holomorphic map. Let $p \in X$ and $v \in T_pX$. Then $\|f_*(v)\|_{C,Y,f(p)} \leq \|v\|_{C,X,p}$ and $\|f_*(v)\|_{K,Y,f(p)} \leq \|v\|_{K,X,p}$;*
(3) *If $X \subset Y$ is a connected open subset and $z \in X$ is a point. Then with any local coordinates we have $BE_Y(z) \leq BE_X(z)$;*
(4) *If the Bergman kernel is positive, then at each point $z \in X$, a peak section σ at z exists. Such a peak section is unique up to a constant factor c with norm 1. Furthermore, with any choice of local coordinates, we have $BE_X(z) = |\sigma(z)|^2$;*
(5) *If the Bergman kernel of X is positive, then $\|\cdot\|_{C,X} \leq 2\|\cdot\|_{B,X}$;*
(6) *If X is a bounded convex domain in $\mathbb{C}^n$, then $\|\cdot\|_{C,X} = \|\cdot\|_{K,X}$;*
(7) *Let $|\cdot|$ be the Euclidean norm and let B_r be the open ball with center 0 and radius r in $\mathbb{C}^n$. Then for any holomorphic tangent vector v at 0,*

$$\|v\|_{C,B_r,0} = \|v\|_{K,B_r,0} = \frac{2}{r}|v|$$

where $|v|$ is the Euclidean norm of v.

Proof. The first six claims are Proposition 4.2.4, Proposition 4.2.3, Proposition 3.5.18, Proposition 4.10.4, Proposition 4.10.3, Theorem 4.10.18 and Theorem 4.8.13 of [6].

The last claim follows from the second claim easily. By rotation, we can assume that $v = b\frac{\partial}{\partial z_1}$. Let Δ_r be the disk with radius r in $\mathbb{C}$ with standard coordinate z and let $\widetilde{v} = b\frac{\partial}{\partial z}$ be the corresponding tangent vector of Δ_r at 0. Now, consider the maps $i: \Delta_r \to B_r$ and $j: B_r \to \Delta_r$ given by $i(z) = (z, 0, \cdots, 0)$ and $j(z_1, \cdots, z_n) = z_1$. We have $i_*(\widetilde{v}) = v$ and $j_*(v) = \widetilde{v}$. By the Schwarz lemma it is easy to see that $\|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|\widetilde{v}|$. So we have

$$\|v\|_{C,B_r,0} \geq \|j_*(v)\|_{C,\Delta_r,0} = \|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|\widetilde{v}| = \frac{2}{r}|v|$$

and

$$\|v\|_{C,B_r,0} = \|i_*(v)\|_{C,B_r,0} \leq \|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|v|.$$

This shows that the last claim holds for the Carathéodory metric. By the sixth claim, we know that the last claim also holds for the Kobayashi metric. This finishes the proof.

□

Now we prove the theorem. We first compare the Carathéodory metric and the Kobayashi metric. By the above lemma it is easy to see that if $X \subset Y$ is a subspace, then $\|\cdot\|_{C,Y} \leq \|\cdot\|_{C,X}$ and $\|\cdot\|_{K,Y} \leq \|\cdot\|_{K,X}$. Let $p \in \mathcal{T}$ be an arbitrary point and let $n = 3g-3 = \dim_{\mathbb{C}} \mathcal{T}$. Let $f_p: \mathcal{T} \to \mathbb{C}^n$ be the Bers' embedding map with $f_p(p) = 0$. In the following, we will identify $\mathcal{T}$ with $f_p(\mathcal{T})$ and $T_p\mathcal{T}$ with $T_0\mathbb{C}^n$. We know that

$$B_2 \subset \mathcal{T} \subset B_6. \tag{2.1}$$

Let $v \in T_0\mathbb{C}^n$ be a holomorphic tangent vector. By using the above lemma we have

$$\|v\|_{C,\mathcal{T},0} \leq \|v\|_{K,\mathcal{T},0} \tag{2.2}$$

and

$$\|v\|_{C,\mathcal{T},0} \geq \|v\|_{C,B_6,0} = \frac{1}{3}|v| = \frac{1}{3}\|v\|_{C,B_2,0} = \frac{1}{3}\|v\|_{K,B_2,0} \geq \frac{1}{3}\|v\|_{K,\mathcal{T},0}. \tag{2.3}$$

By combining the above two inequalities, we have

$$\frac{1}{3}\|v\|_{K,\mathcal{T},0} \leq \|v\|_{C,\mathcal{T},0} \leq \|v\|_{K,\mathcal{T},0}.$$

Since the above constants are independent of the choice of p, we proved the second claim of the theorem.

Now we compare the Bergman metric and the Kobayashi metric. By the above lemma we know that the Bergman norm is bounded from below by half of the Carathéodory norm provided the Bergman kernel is non-zero. For each point $p \in \mathcal{T}_g$, let f_p be the Bers' embedding map with $f_p(p) = 0$. Since $f_p(\mathcal{T}_g) \subset B_6$, by the above lemma we know that $BE_{f_p(\mathcal{T}_g)}(0) \geq BE_{B_6}(0)$. However, we know that the Bergman kernel on B_6 is positive. This implies that the Bergman kernel is non-zero at every point of the Teichmüller space.

By the above lemma and the equivalence of the Carathéodory metric and the Kobayashi metric, we know that the Bergman metric is bounded from below by a constant multiple of the Kobayashi metric.

When we fix a point p and the Bers' embedding map f_p, from inequality (2.3) we know that

$$|v| \leq 3\|v\|_{C,\mathcal{T},0} \leq 3\|v\|_{K,\mathcal{T},0}. \tag{2.4}$$

Let $z_1, \cdots, z_n$ be the standard coordinates on $\mathbb{C}^n$ with $r_i = |z_i|$ and let $dV = \left(\sqrt{-1}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ be the volume form. Let $\sigma = \alpha(z)dz_1 \wedge \cdots \wedge dz_n$ be a peak section over $\mathcal{T}$ at 0 such that

$$\int_{\mathcal{T}} |\alpha|^2 \, dV = 1.$$

Then we have $BE_{\mathcal{T}}(0) = |\alpha(0)|^2$. Now we consider a peak section $\sigma_1 = \alpha_1(z)dz_1 \wedge \cdots \wedge dz_n$ over B_6 at 0 with $\int_{B_6} |\alpha_1|^2 \, dV = 1$. Similarly we have that $BE_{B_6}(0) = |\alpha_1(0)|^2$. By the above lemma and (2.1) we have

$$|\alpha(0)| = (BE_{\mathcal{T}}(0))^{\frac{1}{2}} \geq (BE_{B_6}(0))^{\frac{1}{2}} = |\alpha_1(0)|. \tag{2.5}$$

Let $v_n = \int_{B_1} dV$ be the volume of the unit ball in $\mathbb{C}^n$ and let

$$w_n = \frac{1}{n}\int_{x_1^2+\cdots+x_n^2 \leq 4,\ x_i \geq 0} (x_1^2 + \cdots + x_n^2)x_1 \cdots x_n \, dx_1 \cdots dx_n$$

where $x_1, \cdots, x_n$ are real variables. We see that both v_n and w_n are positive constants only depending on $n = 3g - 3$.

Now we consider the constant section $\sigma_2 = a \, dz_1 \wedge \cdots \wedge dz_n$ over B_6 where $a = 6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}$. we have $\int_{B_6} a^2 \, dV = 1$. Since σ_1 is a peak section at 0, we know that $|\alpha_1(0)| \geq a$. By using inequality (2.5) we have

$$|\alpha(0)| \geq 6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}. \tag{2.6}$$

To estimate the Bergman norm of v, by rotation, we may assume $v = b\frac{\partial}{\partial z_1}$. So $|v| = |b|$. Let $\tau = f(z)dz_1 \wedge \cdots \wedge dz_n$ be an arbitrary section over $\mathcal{T}$ with $f(0) = 0$ and $\int_{\mathcal{T}} |f|^2 \, dV = 1$. We have $\int_{B_2} |f|^2 \, dV \leq 1$.

Let I be the index set $I = \{(i_1, \cdots, i_n) \mid i_k \geq 0, \ \sum i_k \geq 1\}$. Since $f(0) = 0$ and f is holomorphic, we can expand f as a power series on B_2 as

$$f(z) = \sum_{(i_1,\cdots,i_n)\in I} a_{i_1\cdots i_n} z_1^{i_1} \cdots z_n^{i_n}.$$

This implies $df(v) = a_{10\cdots0}b$. Since $\int_{B_2} |f|^2\, dV \le \int_{\mathcal{T}} |f|^2\, dV = 1$, we have

$$1 \ge \int_{B_2} |f|^2\, dV = \int_{B_2} \sum_{(i_1,\cdots,i_n)\in I} |a_{i_1\cdots i_n}|^2 r_1^{2i_1}\cdots r_n^{2i_n}\, dV \ge \int_{B_2} |a_{10\cdots0}|^2 r_1^2\, dV$$

$$= |a_{10\cdots0}|^2 (4\pi)^n \int_{r_1^2+\cdots+r_n^2\le 4} r_1^3 r_2\cdots r_n\, dr_1\cdots dr_n = |a_{10\cdots0}|^2(4\pi)^n w_n$$

which implies

(2.7) $$|a_{10\cdots0}| \le (4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}}.$$

So we have

(2.8) $$|df(v)| = |a_{10\cdots0}||b| \le (4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}} |v|.$$

Let W' be the set of sections over $\mathcal{T}$ such that

$$W' = \{\tau = f(z)dz_1\wedge\cdots\wedge dz_n \mid f(0)=0,\ \int_{\mathcal{T}} |f|^2\, dV = 1\}.$$

By combining (2.4), (2.6) and (2.8) we have

(2.9) $$\begin{aligned} \|v\|_{B,\mathcal{T},0} &= \sup_{\tau\in W'} \frac{|df(v)|}{|\alpha(0)|} \le \frac{(4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}}|v|}{6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}} = \left(\frac{3}{2\pi}\right)^{\frac{n}{2}} \left(\frac{v_n}{w_n}\right)^{\frac{1}{2}} |v| \\ &\le 3\left(\frac{3}{2\pi}\right)^{\frac{n}{2}} \left(\frac{v_n}{w_n}\right)^{\frac{1}{2}} \|v\|_{K,\mathcal{T},0}. \end{aligned}$$

Since the constant in the above inequality only depends on the dimension n, we know that the Bergman metric is uniformly equivalent to the Kobayashi metric. This finished the proof. □

Remark 2.1. After we proved this theorem, the second author was informed by C. McMullen that the equivalence of the Carathéodory metric and the Kobayashi metric maybe known. A more interesting question is whether these two metrics coincide or not. We would like to study this problem in the future.

3. The Negativity of the Ricci Curvature of the Perturbed Ricci Metric

In this section, we first study the curvature bounds of the Ricci metric and the perturbed Ricci metric. By using the Bers' embedding theorem, we show that the injectivity radius of the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric is bounded from below. This implies that both the Ricci metric and the perturbed Ricci metric have bounded geometry on the Teichmüller space.

The boundedness of the curvatures of these metrics was obtained by analyzing their asymptotic behavior. The proof of the negativity of the holomorphic sectional curvature and Ricci curvature of the perturbed Ricci metric is more delicate. Near the boundary of the moduli space and in the degeneration directions, these curvatures are dominated by the contribution from the Ricci metric which is negative. In the nondegeneration directions and in the interior of the moduli space, these curvatures are dominated by the contribution of the constant multiple of the Weil-Petersson metric when the constant is large which is also negative. However we know that the curvature of the linear combinations of two metrics is not linear, we need to handle the error terms carefully.

Let $\mathcal{M}_g$ be the moduli space of closed oriented Riemann surfaces of genus g with $g \ge 2$ and let $\mathcal{T}_g$ be the corresponding Teichmüller space. Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ and let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ be the compactification divisor. It is well known that D is a divisor of normal crossings. In [9] we studied various metrics on $\mathcal{M}_g$ and $\mathcal{T}_g$. We briefly recall the results here.

Fix a point $p \in \mathcal{M}_g$. Let $X = X_p$ be a Riemann surface corresponding to p. Let z be the local holomorphic coordinate on X and let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ where $n = 3g - 3$ is the complex dimension of $\mathcal{M}_g$. Let $HB(X)$ and $Q(X)$ be the spaces of harmonic Beltrami differentials and holomorphic quadratic differentials on X respectively and let λ be the hyperbolic metric on X. Namely,

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda.$$

By the deformation theory of Kodaira-Spencer, we know that the tangent space $T_p\mathcal{M}_g$ is identified with $HB(X)$ and the map $T_p\mathcal{M}_g \to HB(X)$ is given by

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\overline{z}$$

where $A_i = -\partial_{\overline{z}}(\lambda^{-1}\partial_{s_i}\partial_{\overline{z}} \log \lambda)$. Similarly the cotangent space $T_p^*\mathcal{M}_g$ is identified with $Q(X)$. For $\mu = \mu(z)\frac{\partial}{\partial z} \otimes d\overline{z} \in HB(X)$ and $\varphi = \varphi(z)dz^2 \in Q(X)$, the duality between them is given by

$$\langle \mu, \varphi \rangle = \int_X \mu(z)\varphi(z) \ dzd\overline{z}$$

and the Teichmüller norm of φ is defined to be

$$\|\varphi\|_T = \int_X |\varphi(z)| \ dzd\overline{z}.$$

By using the above notation, the norm of the Teichmüller metric is given by

$$\|\mu\|_T = \sup_{\varphi \in Q(X)} \{\mathrm{Re}\langle \mu, \varphi \rangle \mid \|\varphi\|_T = 1\}$$

for all $\mu \in HB(X) \cong T_p\mathcal{M}_g$.

The Weil-Petersson metric on $\mathcal{M}_g$ is defined by

$$h_{i\overline{j}}(p) = \int_X A_i \overline{A_j} \ dv$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form of X. The Ricci metric τ is the negative Ricci curvature of the Weil-Petersson metric

$$\tau_{i\overline{j}} = \partial_i \partial_{\overline{j}} \log \det(h_{k\overline{l}}).$$

By the works in [17] and [9] we know that the Ricci metric is complete. Now we take linear combination of the Ricci metric and the Weil-Petersson metric to define the perturbed Ricci metric

$$\widetilde{\tau}_{i\overline{j}} = \tau_{i\overline{j}} + Ch_{i\overline{j}}$$

where $C > 0$. In [9] we proved the following theorem

Theorem 3.1. *For suitable choice of large constant C, the holomorphic sectional curvature of the perturbed Ricci metric $\widetilde{\tau}$ has negative upper bound. Furthermore, on $\mathcal{M}_g$, the Ricci metric, the perturbed Ricci metric, the Kähler-Einstein metric, the Asymptotic Poincaré metric are equivalent.*

This theorem was proved by using the curvature properties of the perturbed Ricci metric and the estimates of its asymptotic behavior.

Now we prove several claims about the boundedness of the curvature of the Ricci metric and the perturbed Ricci metric which were stated in [9].

Theorem 3.2. *The holomorphic sectional curvature, bisectional curvature and the Ricci curvature of the Ricci metric τ on the moduli space $\mathcal{M}_g$ are bounded.*

As part of the Theorem 4.4 of [9], this theorem was roughly proved in [9]. Here we give a detailed proof since we need the techniques later.

Proof. We follow the notations and computations in [9]. Let $p \in D$ be a codimension m boundary point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates of $\mathcal{M}_g$ at p where $t_1, \cdots, t_m$ represent the degeneration directions. Let $X_{t,s}$ be the Riemann surface corresponding to the point with coordinates $(t_1, \cdots, s_n)$ and let l_i be the length of the short geodesic loop on the i-th collar. Let $u_i = \frac{l_i}{2\pi}$. We fix $\delta > 0$ and assume that $|(t,s)| < \delta$. When δ is small enough, from the work of [18] and [9] we know that

$$u_i = -\frac{\pi}{\log |t_i|}\left(1 + O\left(\left(\frac{\pi}{\log |t_i|}\right)^2\right)\right).$$

Now we let $u_0 = \sum_{i=1}^m u_i + \sum_{j=m+1}^m |s_j|$.

By Corollary 4.2 and Theorem 4.4 of [9], the work of Masur in [12] and Wolpert, if we use $\widetilde{R}_{i\bar{j}k\bar{l}}$ to denote the curvature tensor of the Ricci metric τ, we have

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1 + O(u_0))$, if $i \le m$;
(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j)\right)$, if $i, j \le m$ and $i \ne j$;
(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$, if $i \le m$ and $j \ge m+1$;
(4) $\tau_{i\bar{j}} = O\left(\frac{u_j^2}{|t_j|}\right)$, if $j \le m$ and $i \ge m+1$;
(5) $\tau_{i\bar{j}} = O(1)$, if $i, j \ge m+1$;
(6) The matrix $\left(\tau_{i\bar{j}}\right)_{i,j \ge m+1}$ is positive definite and has a positive lower bound depending on p, n, δ;
(7) $\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0))$, if $i \le m$;
(8) $\widetilde{R}_{i\bar{i}i\bar{i}} = O(1)$, if $i \ge m+1$.

Now we let

$$\Lambda_i = \begin{cases} \frac{u_i}{|t_i|} & i \le m \\ 1 & i \ge m+1. \end{cases}$$

We divide the index set into three parts. Let

(1) $A_1 = \{(i,i,i,i) \mid i \le m\}$;
(2) $A_2 = \{(i,j,k,l) \mid$ at least one of $i,j,k,l \le m$ and they are not all equal$\}$;
(3) $A_3 = \{(i,j,k,l) \mid i,j,k,l \ge m+1\}$.

By following the computations of [9] we know that, if $(i,j,k,l) \in A_2$, then

$$\widetilde{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0).$$

Let $v = a_1 \frac{\partial}{\partial t_1} + \cdots + a_n \frac{\partial}{\partial s_n}$ and $w = b_1 \frac{\partial}{\partial t_1} + \cdots + b_n \frac{\partial}{\partial s_n}$ be two tangent vectors at (t,s). We have

$$|\widetilde{R}(v, \overline{v}, w, \overline{w})| = \left|\sum_{i,j,k,l} a_i \overline{a}_j b_k \overline{b}_l \widetilde{R}_{i\bar{j}k\bar{l}}\right| \le \sum_{i,j,k,l}\left|a_i \overline{a}_j b_k \overline{b}_l \widetilde{R}_{i\bar{j}k\bar{l}}\right| = I_1 + I_2 + I_3$$

where $I_\alpha = \sum_{(i,j,k,l)\in A_\alpha} \left| a_i\overline{a}_j b_k\overline{b}_l \widetilde{R}_{i\overline{j}k\overline{l}} \right|$. To estimate the norms of v and w, we have

$$\tau(v,v) = \sum_{i,j\le m, i\ne j} a_i\overline{a}_j\tau_{i\overline{j}} + \sum_{i\le m<j} a_i\overline{a}_j\tau_{i\overline{j}} + \sum_{j\le m<i} a_i\overline{a}_j\tau_{i\overline{j}} + \sum_{i\le m} |a_i|^2\tau_{i\overline{i}} + \sum_{i,j\ge m+1} a_i\overline{a}_j\tau_{i\overline{j}}.$$

By using the asymptotic of τ and the Schwarz inequality we have

$$\left| \sum_{i,j\le m, i\ne j} a_i\overline{a}_j\tau_{i\overline{j}} + \sum_{i\le m<j} a_i\overline{a}_j\tau_{i\overline{j}} + \sum_{j\le m<i} a_i\overline{a}_j\tau_{i\overline{j}} \right| \le O(u_0)\sum_{i=1}^n |a_i|^2\Lambda_i^2.$$

Since the matrix $\left(\tau_{i\overline{j}}\right)_{i,j\ge m+1}$ has a local positive lower bound, we know there is a positive constant c depending on p,n,δ such that

$$\sum_{i,j\ge m+1} a_i\overline{a}_j\tau_{i\overline{j}} \ge c\sum_{i=1}^n |a_i|^2 = c\sum_{i=m+1}^n |a_i|^2\Lambda_i^2.$$

Finally we have

$$\sum_{i\le m} |a_i|^2\tau_{i\overline{i}} = \frac{3}{4\pi^2}(1+O(u_0))\sum_{i=1}^m |a_i|^2\Lambda_i^2.$$

By combining the above inequalities we know there is another positive constant c_1 depending on p,n,δ such that

$$\tau(v,v) \ge c_1\sum_{i=1}^n |a_i|^2\Lambda_i^2. \tag{3.1}$$

Similar estimates hold for the Ricci norm of w.

Now for each term in I_2, by using the Schwarz inequality, we have

$$\left| a_i\overline{a}_j b_k\overline{b}_l \widetilde{R}_{i\overline{j}k\overline{l}} \right| = O(u_0)|a_i\overline{a}_j b_k\overline{b}_l\Lambda_i\Lambda_j\Lambda_k\Lambda_l| \le O(u_0)(|a_i|^2\Lambda_i^2 + |a_j|^2\Lambda_j^2)(|b_k|^2\Lambda_k^2 + |b_l|^2\Lambda_l^2).$$

So we have

$$I_2 = O(u_0)\left(\sum_{i=1}^n |a_i|^2\Lambda_i^2\right)\left(\sum_{i=1}^n |b_i|^2\Lambda_i^2\right) \le c_0\tau(v,v)\tau(w,w)$$

for some positive constant c_0. By enlarging this constant, we also have

$$I_3 \le O(1)\sum_{(i,j,k,l)\in A_3} |a_i\overline{a}_j b_k\overline{b}_l| \le O(1)\left(\sum_{i=m+1}^n |a_i|^2\Lambda_i^2\right)\left(\sum_{i=m+1}^n |b_i|^2\Lambda_i^2\right) \le c_0\tau(v,v)\tau(w,w)$$

and

$$I_1 = \frac{3}{8\pi^4}(1+O(u_0))\sum_{i=1}^m |a_i|^2|b_i|^2\Lambda_i^4 \le c_0\tau(v,v)\tau(w,w).$$

By combining the above inequalities we know that there is a positive constant $\widetilde{c}$ depending on p,n,δ such that if δ is small enough, then

$$|\widetilde{R}(v,\overline{v},w,\overline{w})| \le \widetilde{c}\tau(v,v)\tau(w,w).$$

So we have proved that for each point $p\in D$ there is an open neighborhood U_p such that the bisectional curvature of the Ricci metric is bounded by a constant which depends on U_p. Since D is compact, we can find a finite cover of D by such U_p. Let U be the union of such U_p. Then we can find a universal constant c which bounds the bisectional curvature at each point in U. Since $\mathcal{M}_g \setminus U$ is a compact set, we know the bisectional curvature is bounded there. So we proved that the bisectional curvature of the Ricci metric is bounded.

The boundedness of the holomorphic sectional curvature can be proved similarly if we replace w by v in the above argument. Finally since the Ricci curvature is the average of the bisectional curvature and the holomorphic sectional curvature, it is bounded. We finish the proof.

$\square$

We now investigate the curvatures of the perturbed Ricci metric. We have

Theorem 3.3. *For any constant $C > 0$, the bisectional curvature of the perturbed Ricci metric $\widetilde{\tau} = \tau + Ch$ is bounded. Furthermore, with suitable choice of C, the holomorphic sectional curvature and the Ricci curvature of $\widetilde{\tau}$ are bounded from above and below by negative constants.*

Proof. We use $R_{i\bar{j}k\bar{l}}$ and $P_{i\bar{j}k\bar{l}}$ to denote the curvature tensor of the Weil-Petersson metric and the perturbed Ricci metric respectively. We use the same notations as in the proof of the above theorem. Let $p \in D$ be a codimension m boundary point, let $(t_1, \cdots, s_n)$ be the local pinching coordinates and let A_1, A_2, A_3 be the partition of the index set. Assume δ is a small positive constant and $|(t,s)| < \delta$.

By Corollary 4.1, Corollary 4.2 and Theorem 5.2 of [9] we have

(1) $\widetilde{\tau}_{i\bar{i}} = \frac{u_i^2}{|t_i|^2}\left(\frac{3}{4\pi^2} + \frac{1}{2}Cu_i\right)(1 + O(u_0))$, if $i \le m$;
(2) $\widetilde{\tau}_{i\bar{j}} = \frac{u_i^2 u_j^2}{|t_i t_j|}(O(u_i + u_j) + CO(u_i u_j))$, if $i, j \le m$ and $i \ne j$;
(3) $\widetilde{\tau}_{i\bar{j}} = \frac{u_i^2}{|t_i|}(O(1) + CO(u_i))$, if $i \le m$ and $j \ge m+1$;
(4) $\widetilde{\tau}_{i\bar{j}} = \frac{u_j^2}{|t_j|}(O(1) + CO(u_j))$, if $j \le m$ and $i \ge m+1$;
(5) $P_{i\bar{i}i\bar{i}} = \left(\left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4}\left(1 + \frac{2\pi^2 C u_i}{3}\right)^{-1}\right)\frac{u_i^4}{|t_i|^4} + \frac{3C}{8\pi^2}\frac{u_i^5}{|t_i|^4}\right)(1 + O(u_0))$, if $i \le m$;
(6) $P_{i\bar{i}i\bar{i}} = O(1) + CR_{i\bar{i}i\bar{i}}$, if $i \ge m+1$;
(7) $P_{i\bar{j}k\bar{l}} = O(1) + CR_{i\bar{j}k\bar{l}}$, if $(i,j,k,l) \in A_3$;
(8) $P_{i\bar{j}k\bar{l}} = O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0) + CR_{i\bar{j}k\bar{l}}$, if $(i,j,k,l) \in A_2$

where all the O-terms are independent of C.

Let v and w be holomorphic vectors as above. To estimate the bisectional curvature, we have

$$
\begin{aligned}
|P(v,\overline{v},w,\overline{w})| \le & \sum_{i=1}^m |a_i|^2|b_i|^2 P_{i\bar{i}i\bar{i}} + \sum_{(i,j,k,l)\in A_2}\left|a_i\overline{a}_j b_k\overline{b}_l P_{i\bar{j}k\bar{l}}\right| + \sum_{(i,j,k,l)\in A_3}\left|a_i\overline{a}_j b_k\overline{b}_l P_{i\bar{j}k\bar{l}}\right| \\
\le & \sum_{i=1}^m |a_i|^2|b_i|^2 P_{i\bar{i}i\bar{i}} + O(u_0)\sum_{(i,j,k,l)\in A_2}\left|a_i\overline{a}_j b_k\overline{b}_l\Lambda_i\Lambda_j\Lambda_k\Lambda_l\right| \\
& + C\sum_{(i,j,k,l)\in A_2}\left|a_i\overline{a}_j b_k\overline{b}_l R_{i\bar{j}k\bar{l}}\right| + O(1)\sum_{(i,j,k,l)\in A_3}\left|a_i\overline{a}_j b_k\overline{b}_l\right| \\
& + C\sum_{(i,j,k,l)\in A_3}\left|a_i\overline{a}_j b_k\overline{b}_l R_{i\bar{j}k\bar{l}}\right|.
\end{aligned}
\tag{3.2}
$$

Let c_i denote certain positive constants only depending on p, n, δ. By the proof of the above theorem, since $\widetilde{\tau} \ge \tau$, we have

$$
\left|O(u_0)\sum_{(i,j,k,l)\in A_2}\left|a_i\overline{a}_j b_k\overline{b}_l\Lambda_i\Lambda_j\Lambda_k\Lambda_l\right|\right| \le c_0\tau(v,v)\tau(w,w) \le c_0\widetilde{\tau}(v,v)\widetilde{\tau}(w,w). \tag{3.3}
$$

We also have

$$
\left|O(1)\sum_{(i,j,k,l)\in A_3}\left|a_i\overline{a}_j b_k\overline{b}_l\right|\right| \le c_1\tau(v,v)\tau(w,w) \le c_2\widetilde{\tau}(v,v)\widetilde{\tau}(w,w). \tag{3.4}
$$

Now we estimate $\widetilde{\tau}(v,v)$. By using similar argument as in the above proof we know that

$$\widetilde{\tau}(v,v) \geq c_2 \sum_{i=1}^{n} |a_i|^2 \widetilde{\tau}_{i\bar{i}}.$$

So for $i \leq m$ we have

$$|a_i|^2|b_i|^2 P_{i\bar{i}i\bar{i}} = |a_i|^2|b_i|^2 \left(\left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4}\left(1+\frac{2\pi^2 C u_i}{3}\right)^{-1}\right) \frac{u_i^4}{|t_i|^4} + \frac{3C}{8\pi^2}\frac{u_i^5}{|t_i|^4}\right)(1+O(u_0))$$

$$\leq c_3 |a_i|^2|b_i|^2 \frac{u_i^4}{|t_i|^4}\left(\frac{3}{4\pi^2}+\frac{1}{2}Cu_i\right)^2 \leq c_4 |a_i|^2|b_i|^2 \widetilde{\tau}_{i\bar{i}}^2$$

which implies

$$\sum_{i=1}^{m} |a_i|^2|b_i|^2 P_{i\bar{i}i\bar{i}} \leq c_5 \sum_{i=1}^{n} |a_i|^2|b_i|^2 \widetilde{\tau}_{i\bar{i}}^2 \leq c_6 \widetilde{\tau}(v,v)\widetilde{\tau}(w,w). \tag{3.5}$$

To estimate the rest two terms in the right hand side of (3.2) we need the estimates of the curvature tensor of the Weil-Petersson metric which is done in the proof of Corollary 4.2 of [9]. By collecting the results there we know that $R_{i\bar{j}k\bar{l}} = O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0)$ if $(i,j,k,l) \in A_2$ and $R_{i\bar{j}k\bar{l}} = O(1)$ if $(i,j,k,l) \in A_3$. By using a similar argument as in the above proof we know that

$$\left| C \sum_{(i,j,k,l)\in A_2} \left| a_i \overline{a}_j b_k \overline{b}_l R_{i\bar{j}k\bar{l}} \right| \right| \leq c_7 C \widetilde{\tau}(v,v)\widetilde{\tau}(w,w) \tag{3.6}$$

and

$$\left| C \sum_{(i,j,k,l)\in A_3} \left| a_i \overline{a}_j b_k \overline{b}_l R_{i\bar{j}k\bar{l}} \right| \right| \leq c_8 C \widetilde{\tau}(v,v)\widetilde{\tau}(w,w). \tag{3.7}$$

These imply that

$$|P(v,\overline{v},w,\overline{w})| \leq (c_9 C + c_{10})\widetilde{\tau}(v,v)\widetilde{\tau}(w,w).$$

By using the compactness argument as above we proved that the bisectional curvature of $\widetilde{\tau}$ is bounded. However, the bounds depend on the choice of C.

By using a similar method it is easy to see that the holomorphic sectional curvature is also bounded. However, in [9] we showed that, for suitable choice of C, the holomorphic sectional curvature has a negative upper bound. So for this C, the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched between negative constants.

Finally, we consider the Ricci curvature of $\widetilde{\tau} = \tau + C\,h$. We first define two new tensors. Let $\widehat{R}_{i\bar{j}k\bar{l}} = P_{i\bar{j}k\bar{l}} - CR_{i\bar{j}k\bar{l}}$ and let $P_{i\bar{j}} = -Ric(\omega_{\widetilde{\tau}})_{i\bar{j}}$. We only need to show that there are positive constants α_1 and α_2 which may depend on C such that

$$\alpha_1 \left(\widetilde{\tau}_{i\bar{j}}\right) \leq \left(P_{i\bar{j}}\right) \leq \alpha_2 \left(\widetilde{\tau}_{i\bar{j}}\right). \tag{3.8}$$

Based on Lemma 5.2 of [9] and by Corollary 4.1 and 4.2 of [9] we can estimate the asymptotic of the perturbed Ricci metric.

Lemma 3.1. *Let $p \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ be a codimension m boundary point and let $(t,s) = (t_1,\cdots,s_n)$ be the pinching coordinates. Let $\delta > 0$ be a small constant such that $|(t,s)| < \delta$. Let C be a positive constant. Let $B_1 = \left(\tau_{i\bar{j}}\right)_{i,j\geq m+1}$, let $B_2 = \left(h_{i\bar{j}}\right)_{i,j\geq m+1}$ and let $B = B_1 + CB_2$. Let $\left(B^{i\bar{j}}\right) = \overline{(B^{-1})}$ and let $x_i = 2\pi^2 C u_i$ for $i \leq m$. Then we have*

(1) $\widetilde{\tau}_{i\bar{i}} = \frac{\Lambda_i^2}{4\pi^2}(3+x_i)(1+O(u_0))$ *and* $\widetilde{\tau}^{i\bar{i}} = 4\pi^2\Lambda_i^{-2}(3+x_i)^{-1}(1+O(u_0))$ *if* $i \leq m$;

(2) $\widetilde{\tau}_{i\bar{j}} = \frac{u_i^2 u_j^2}{|t_i t_j|}(O(u_i+u_j)+CO(u_iu_j))$ *and* $\widetilde{\tau}^{i\bar{j}} = O(|t_it_j|)\min\{(1+x_i)^{-1},(1+x_j)^{-1}\}$ *if* $i,j\le m$ *and* $i\neq j$*;*

(3) $\widetilde{\tau}_{i\bar{j}} = \frac{u_i^2}{|t_i|}(O(1)+CO(u_i))$ *and* $\widetilde{\tau}^{i\bar{j}} = O(|t_i|)(1+x_i)^{-1}$ *if* $i\le m$ *and* $j\ge m+1$*;*

(4) $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ *and* $\widetilde{\tau}^{i\bar{j}} = C^{-1}\left(h^{i\bar{j}}+O(C^{-1})+O(u_0)\right)$ *if* $i,j\ge m+1$*;*

(5) $\widehat{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0)$ *if* $(i,j,k,l)\in A_2$*;*

(6) $\widehat{R}_{i\bar{j}k\bar{l}} = O(1)$ *if* $(i,j,k,l)\in A_3$.

Proof. Let $a=a(t_1,\bar{t}_1,\cdots,s_n,\bar{s}_n)$ and $b=b(t_1,\bar{t}_1,\cdots,s_n,\bar{s}_n)$ be any local functions defined for $|(t,s)|<\delta$. Assume there is local constant c_1 depending on p,δ and n such that

$$0<c_1\le a,b$$

for $|(t,s)|<\delta$. We first realize that there are constants $\mu_i>0$ depending on c_1, p, n and δ such that

$$1+x_i\le \mu_i(a+Cb)$$

for $|(t,s)|<\delta$. In fact, we can pick $\mu_i=\max\{\frac{1}{c_1},\frac{2\pi^2u_i}{c_1}\}$ since u_i is small when δ is small.

The first four claims followed from Corollary 4.1, Corollary 4.2, Lemma 5.1 and Lemma 5.2 of [9]. By the proof of Lemma 5.2 of [9] we have the linear algebraic formula

$$\det(\widetilde{\tau}) = \left(\prod_{i=1}^m \frac{\Lambda_i^2}{4\pi^2}(3+x_i)\right)\det(B)(1+O(u_0)).$$

These claims followed from similar computations of the determinants of the minor matrices.

The last two claims follow from the same techniques and computations as in the appendix of [9].

□

Now we estimate $P_{i\bar{j}}$. We first compute $P_{i\bar{i}}$ with $i\le m$. We have

$$\begin{aligned} P_{i\bar{i}} =&\widetilde{\tau}^{i\bar{i}}P_{i\bar{i}i\bar{i}} + \sum_{(k,l)\neq(i,i)}\widetilde{\tau}^{k\bar{l}}P_{i\bar{i}k\bar{l}} \\ =&\widetilde{\tau}^{i\bar{i}}P_{i\bar{i}i\bar{i}} + \sum_{(k,l)\neq(i,i)}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{i}k\bar{l}} + C\sum_{(k,l)\neq(i,i)}\widetilde{\tau}^{k\bar{l}}R_{i\bar{i}k\bar{l}}. \end{aligned}$$

We estimate each term in the right hand side of the above formula. We have

$$\begin{aligned} \widetilde{\tau}^{i\bar{i}}P_{i\bar{i}i\bar{i}} =&4\pi^2\Lambda_i^{-2}(3+x_i)^{-1}\left(\left(\frac{9}{16\pi^4}-\frac{3}{16\pi^4}\left(1+\frac{x_i}{3}\right)^{-1}\right)\Lambda_i^4+\frac{3x_i}{16\pi^4}\Lambda_i^4\right)(1+O(u_0)) \\ =&\frac{3}{4\pi^2}\Lambda_i^2(3+x_i)^{-1}\left(3-\left(1+\frac{x_i}{3}\right)^{-1}+x_i\right)(1+O(u_0)) \\ =&\frac{3}{4\pi^2}\left(1-\frac{3}{(3+x_i)^2}\right)\Lambda_i^2(1+O(u_0)). \end{aligned} \tag{3.9}$$

By the fifth claim of the above lemma we have

$$\left|\sum_{(k,l)\neq(i,i)}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{i}k\bar{l}}\right| = O(\Lambda_i^2)O(u_0). \tag{3.10}$$

We also have

$$C\sum_{(k,l)\neq(i,i)}\widetilde{\tau}^{k\bar{l}}R_{i\bar{i}k\bar{l}} = C\sum_{k\neq i,\ l\neq i}\widetilde{\tau}^{k\bar{l}}R_{i\bar{i}k\bar{l}} + C\sum_{k\neq i}\widetilde{\tau}^{k\bar{i}}R_{i\bar{i}k\bar{i}} + C\sum_{l\neq i}\widetilde{\tau}^{i\bar{l}}R_{i\bar{i}i\bar{l}}.$$

By the proof of Corollary 4.2 of [9] we know that, if $k \neq i$, then $R_{i\bar{i}k\bar{i}} = O\left(\frac{u_i^5}{|t_i|^3}\right) O(\Lambda_k)$. By combining with the above lemma, we have

$$\left| C \sum_{k \neq i} \widetilde{\tau}^{k\bar{i}} R_{i\bar{i}k\bar{i}} \right| = CO(u_i^3)\Lambda_i^2(1+x_i)^{-1} = \frac{x_i}{1+x_i}\Lambda_i^2 O(u_i^2) = \Lambda_i^2 O(u_i^2). \tag{3.11}$$

Similarly, we have

$$\left| C \sum_{l \neq i} \widetilde{\tau}^{i\bar{l}} R_{i\bar{i}i\bar{l}} \right| = \Lambda_i^2 O(u_i^2). \tag{3.12}$$

Now we fix i and let $v = \frac{\partial}{\partial t_i}$, $w = b_1 \frac{\partial}{\partial t_1} + \cdots + b_n \frac{\partial}{\partial s_n}$ with $b_i = 0$. Since the bisectional curvature of the Weil-Petersson metric is non-positive, we have

$$0 \leq R(v, \overline{v}, w, \overline{w}) = \sum_{k \neq i,\ l \neq i} b_k \overline{b}_l R_{i\bar{i}k\bar{l}}.$$

This implies that the matrix $\left(R_{i\bar{i}k\bar{l}}\right)_{k \neq i,\ l \neq i}$ is semi-positive definite. So we know

$$\sum_{k \neq i,\ l \neq i} \widetilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} \geq 0$$

since it is the trace of the product of a positive definite matrix and a semi-positive definite matrix. Again, by using the proof of Lemma 4.2 of [9] we have

$$0 \leq C \sum_{k \neq i,\ l \neq i} \widetilde{\tau}^{k\bar{l}} R_{i\bar{i}k\bar{l}} \leq x_i \Lambda_i^2 O^+(1) \tag{3.13}$$

where $O^+(1)$ represents a positive bounded term. By combining formulas (3.9), (3.10), (3.11), (3.12) and (3.13) we have

$$P_{i\bar{i}} \geq \frac{3}{4\pi^2}\left(1 - \frac{3}{(3+x_i)^2}\right)\Lambda_i^2(1+O(u_0)) + \Lambda_i^2 O(u_0) + \Lambda_i^2 O(u_i^2)$$

and

$$P_{i\bar{i}} \leq \frac{3}{4\pi^2}\left(1 - \frac{3}{(3+x_i)^2} + x_i O^+(1)\right)\Lambda_i^2(1+O(u_0)) + \Lambda_i^2 O(u_0) + \Lambda_i^2 O(u_i^2)$$

which imply

$$\frac{3\Lambda_i^2}{8\pi^2}\left(1 - \frac{3}{(3+x_i)^2}\right) \leq P_{i\bar{i}} \leq \frac{3\Lambda_i^2}{2\pi^2}\left(1 - \frac{3}{(3+x_i)^2} + x_i O^+(1)\right) \tag{3.14}$$

when δ is small. The above estimate is independent of the choice of C.

Now we estimate $P_{i\bar{j}}$ with $i, j \leq m$ and $i \neq j$. We have

$$\begin{aligned} P_{i\bar{j}} =& \widetilde{\tau}^{k\bar{l}} P_{i\bar{j}k\bar{l}} = \widetilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C\widetilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\ =& \widetilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} + C \sum_{k \leq m} \widetilde{\tau}^{k\bar{k}} R_{i\bar{j}k\bar{k}} + C \sum_{k,l \leq m,\ k \neq l} \widetilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} \\ &+ C \sum_{k \leq m < l} \widetilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{l \leq m < k} \widetilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}} + C \sum_{k,l \geq m+1} \widetilde{\tau}^{k\bar{l}} R_{i\bar{j}k\bar{l}}. \end{aligned} \tag{3.15}$$

By the above lemma we have

$$\left| \widetilde{\tau}^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \right| = \Lambda_i \Lambda_j O(u_0). \tag{3.16}$$

We also have

$$\left|C\sum_{k\le m}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}\right| \le \left|C\sum_{k\le m,k\ne i,j}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}\right| + \left|C\widetilde{\tau}^{i\bar{i}}R_{i\bar{j}i\bar{i}}\right| + \left|C\widetilde{\tau}^{j\bar{j}}R_{i\bar{j}j\bar{j}}\right|.$$

By the proof of Lemma 4.2 of [9] we have $R_{i\bar{j}i\bar{i}} = O\left(\frac{u_i^5u_j^3}{|t_i^3t_j|}\right)$ which implies

$$\left|C\widetilde{\tau}^{i\bar{i}}R_{i\bar{j}i\bar{i}}\right| = 4\pi^2\Lambda_i^{-2}(3+x_i)^{-1}CO\left(\frac{u_i^5u_j^3}{|t_i^3t_j|}\right) = \Lambda_i\Lambda_jO(u_0).$$

Similarly we have

$$\left|C\widetilde{\tau}^{j\bar{j}}R_{i\bar{j}j\bar{j}}\right| = \Lambda_i\Lambda_jO(u_0).$$

Again, by the proof of Lemma 4.2 of [9], for $k\le m$ and $k\ne i,j$ we have

$$R_{i\bar{j}k\bar{k}} = O\left(\frac{u_iu_ju_k^3}{|t_it_jt_k^2|}u_0\right)$$

which implies

$$\left|C\sum_{k\le m,k\ne i,j}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}\right| = \Lambda_i\Lambda_jO(u_0).$$

By combining the above three formulas we have

$$\left|C\sum_{k\le m}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}\right| = \Lambda_i\Lambda_jO(u_0).$$

Similarly we can show that

$$\left|C\sum_{k,l\le m,\ k\ne l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right| = \Lambda_i\Lambda_jO(u_0),$$

$$\left|C\sum_{k\le m<l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right| = \Lambda_i\Lambda_jO(u_0)$$

and

$$\left|C\sum_{l\le m<k}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right| = \Lambda_i\Lambda_jO(u_0).$$

Finally,

$$\left|C\sum_{k,l\ge m+1}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right| \le \sum_{k,l\ge m+1}\left|C\widetilde{\tau}^{k\bar{l}}\right|\left|R_{i\bar{j}k\bar{l}}\right| = \sum_{k,l\ge m+1}O(1)\left|R_{i\bar{j}k\bar{l}}\right| = \Lambda_i\Lambda_jO(u_0).$$

By combining the above results we have

(3.17) $$P_{i\bar{j}} = \Lambda_i\Lambda_jO(u_0).$$

By using the same method we know that, if $i\le m<j$, then

(3.18) $$P_{i\bar{j}} = \Lambda_iO(u_0)$$

and if $j\le m<i$, then

(3.19) $$P_{i\bar{j}} = \Lambda_jO(u_0).$$

The next step is to estimate the matrix $\left(P_{i\bar{j}}\right)_{i,j\geq m+1}$. We will show that this matrix is bounded from above and below by positive constant multiples of the matrix B_1 defined in the above lemma where the constants depend on δ and C. We first estimate $P_{i\bar{j}}$ with fixed $i,j\geq m+1$. We have

$$\begin{aligned}P_{i\bar{j}} =&\widetilde{\tau}^{k\bar{l}}P_{i\bar{j}k\bar{l}}=\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}+\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\\ =&\sum_{k\leq m}\widetilde{\tau}^{k\bar{k}}\widehat{R}_{i\bar{j}k\bar{k}}+\sum_{k,l\leq m,\ k\neq l}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}+\sum_{k\leq m<l}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}+\sum_{l\leq m<k}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}\\ &+\sum_{k,l\geq m+1}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}+C\sum_{k\leq m}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}+C\sum_{k,l\leq m,\ k\neq l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\\ &+C\sum_{k\leq m<l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}+C\sum_{l\leq m<k}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}+C\sum_{k,l\geq m+1}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}.\end{aligned}$$

By Lemma 3.1 and the proof of Corollary 4.2 of [9] we know that $\left|\sum_{k\leq m}\widetilde{\tau}^{k\bar{k}}\widehat{R}_{i\bar{j}k\bar{k}}\right|=O(u_0)$, $\left|\sum_{k,l\leq m,\ k\neq l}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}\right|=O(u_0)$, $\left|\sum_{k\leq m<l}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}\right|=O(u_0)$, $\left|\sum_{l\leq m<k}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}\right|=O(u_0)$, $\left|\sum_{k,l\geq m+1}\widetilde{\tau}^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}}\right|=O(C^{-1})$, $\left|C\sum_{k,l\leq m,\ k\neq l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right|=O(u_0)$, $\left|C\sum_{k\leq m<l}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right|=O(u_0)$ and $\left|C\sum_{l\leq m<k}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right|=O(u_0)$. Also, since for $i,j,k,l\geq m+1$, $R_{i\bar{j}k\bar{l}}=O(1)$, we have

$$C\sum_{k,l\geq m+1}\widetilde{\tau}^{k\bar{l}}R_{i\bar{j}k\bar{l}}=\sum_{k,l\geq m+1}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}+O(C^{-1})+O(u_0).$$

By combining the above arguments we have

$$P_{i\bar{j}}=C\sum_{k\leq m}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}+\sum_{k,l\geq m+1}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}+O(C^{-1})+O(u_0). \tag{3.20}$$

The matrix $\left(\sum_{k,l\geq m+1}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}(0,s)\right)_{i,j\geq m+1}$ is just the negative of the Ricci curvature matrix of the restriction of the Weil-Petersson metric to the boundary piece. So we know it is positive definite and is bounded from below by a constant multiple of $B_2(0,s)$. By continuity we know that, when δ is small enough, the matrix $\left(\sum_{k,l\geq m+1}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}(t,s)\right)_{i,j\geq m+1}$ is bounded from below by a constant multiple of $B_2(t,s)$. Again, since $h^{k\bar{l}}R_{i\bar{j}k\bar{l}}=O(1)$ when $i,j,k,l\geq m+1$ and the fact that matrices B_1 and B_2 are locally equivalent, we know that $\left(\sum_{k,l\geq m+1}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}\right)_{i,j\geq m+1}$ is locally bounded from above and below by positive constants multiples of B_1.

Finally, by using the fact that the bisectional curvature of the Weil-Petersson metric is non-positive and $\widetilde{\tau}^{k\bar{k}}>0$, we know that the matrix $\left(\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}\right)_{i,j\geq m+1}$ is positive semi-definite. Also, we know that $C\sum_{k\leq m}\widetilde{\tau}^{k\bar{k}}R_{i\bar{j}k\bar{k}}=O(1)$.

Now by using formula (3.20) we know that there are positive constants $\beta_1\leq\beta_2$ depending on δ, the point p and the choice of C such that as long as δ is small enough and C is large enough,

$$\beta_1B_1\leq\left(P_{i\bar{j}}\right)_{i,j\geq m+1}\leq\beta_2B_1. \tag{3.21}$$

We know that there is a constant $c_0>0$ such that $h\leq c_0\tau$ which implies $\tau\leq\widetilde{\tau}\leq(1+c_0C)\tau$. By combining formulas (3.14), (3.17), (3.18), (3.19) and (3.21) we know that, when δ is small enough and C is large enough, there are positive constants $\alpha_1\leq\alpha_2$ depending on p, δ and C such that

$$\alpha_1\widetilde{\tau}\leq\left(P_{i\bar{j}}\right)\leq\alpha_2\widetilde{\tau}.$$

Now by using the compactness argument as we did before, we can find an open neighborhood U of D in $\overline{\mathcal{M}}_g$ and a $C_0 > 0$ such that

$$\alpha_1 \widetilde{\tau} \leq \left(P_{i\bar{j}} \right) \leq \alpha_2 \widetilde{\tau}$$

on U for positive constants α_1 and α_2 as long as $C \geq C_0$.

Let $V = \overline{\mathcal{M}}_g \setminus U$. We know V is compact. We also know that, for C large enough,

$$Ric(\widetilde{\tau}) = Ric(C^{-1}\widetilde{\tau}) = Ric(h + C^{-1}\tau).$$

Since the Ricci curvature of the Weil-Petersson metric has a negative upper bound, a perturbation of the Weil-Petersson metric with a small error term still has negative Ricci curvature on a compact set V. Also, on V the perturbed Ricci metric is bounded. So for C large, we know that the Ricci curvature of the perturbed Ricci metric is pinched between negative constant multiples of the perturbed Ricci metric. Here the bounds depend on the choice of C. This finished the proof.

□

As a direct corollary of the above theorem, we show a vanishing theorem similar to the work of Faltings [2].

Corollary 3.1. *Let D be the compactification divisor of the Deligne-Mumford compactification of $\mathcal{M}_g$. Then*

$$H^0 \left(\overline{\mathcal{M}}_g, \Omega\left(\log D\right)^* \right) = 0.$$

Proof. We first pick a constant $C > 0$ such that the Ricci curvature of the perturbed Ricci metric $\widetilde{\tau} = \tau + Ch$ is pinched by negative constants. Let σ be a holomorphic section of $\Omega\left(\log D\right)^*$.

Let $p \in \overline{\mathcal{M}_g} \backslash \mathcal{M}_g$ be a codimension m point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be local pinching coordinates. Then locally we have

$$\sigma = \sum_{i=1}^{m} a_i(t,s) t_i \frac{\partial}{\partial t_i} + \sum_{j=m+1}^{n} a_j(t,s) \frac{\partial}{\partial s_j}$$

where a_i are bounded local holomorphic functions for $1 \leq i \leq n$. It is clear that, restricted to $\mathcal{M}_g$, σ is a holomorphic vector field. Now we equip the moduli space $\mathcal{M}_g$ with the perturbed Ricci metric $\widetilde{\tau}$. From the above expression of σ, it is easy to see that

$$\|\sigma\|_{\widetilde{\tau}} \in L^2(\mathcal{M}_g, \widetilde{\tau})$$

since $\widetilde{\tau}$ is equivalent to the asymptotic Poincaré metric. Now we have the Bochner formula

$$\Delta_{\widetilde{\tau}} \|\sigma\|_{\widetilde{\tau}}^2 = \|\nabla \sigma\|_{\widetilde{\tau}}^2 - Ric_{\widetilde{\tau}}(\sigma, \sigma).$$

To integrate, we need a special cut-off function. In [10], a monotone sequence of cut-off functions ρ_ϵ with the properties that $\Delta_{\widetilde{\tau}} \rho_\epsilon$ is uniformly bounded for each ϵ and the measure of the support of $\Delta_{\widetilde{\tau}} \rho_\epsilon$ goes to 0 as ϵ goes to zero. We will recall the construction in the next section.

By using the cut-off function ρ_ϵ we have

$$\lim_{\epsilon \to 0} \int_{\mathcal{M}_g} \rho_\epsilon \Delta \|\sigma\|_{\widetilde{\tau}}^2 dV_{\widetilde{\tau}} = \lim_{\epsilon \to 0} \int_{\mathcal{M}_g} \Delta \rho_\epsilon \|\sigma\|_{\widetilde{\tau}}^2 dV_{\widetilde{\tau}} = 0$$

since σ is an L^2 section with respect to $\widetilde{\tau}$ and the measure of $\Delta_{\widetilde{\tau}} \rho_\epsilon$ goes to 0. This above formula implies $Ric(\sigma, \sigma) = 0$ since $Ric(\widetilde{\tau})$ is negative which implies $\sigma = 0$. Thus we have proved the corollary.

□

Finally, we show that the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric has bounded geometry.

Corollary 3.2. *The injectivity radius of the Teichmüller space equipped with the Ricci metric or the perturbed Ricci metric is bounded from below.*

Proof. We only prove that there is a lower bound for the injectivity radius of the Ricci metric since the case of the perturbed Ricci metric can be done in the same way.

In Theorem 3.2 we showed that the curvature of the Ricci metric is bounded. We denote the sup of the curvature by δ. If $\delta \leq 0$ then the injectivity radius is $+\infty$ by the Cartan-Hadamard theorem. Now we assume $\delta > 0$.

Assume the injectivity radius of $(\mathcal{T}_g, \tau)$ is 0, then for any $\epsilon > 0$, there is a point $p = p_\epsilon$ such that the injectivity radius at p is less than ϵ.

Let f_p be the Bers' embedding map such that $f_p(p) = 0$. By using a similar argument as in the proof of Theorem 2.1, and by changing some constants, we know that the Ricci metric and the Euclidean metric are equivalent on the Euclidean ball $B_1 \subset f_p(\mathcal{T}_g)$. By using the Rauch comparison theorem to compare the Ricci metric on the ball B_1 and the standard sphere of constant curvature δ, we know that there is no conjugate point of p within distance ϵ when ϵ is small enough.

So the only case we need to rule out is that there is a closed geodesic loop γ containing p such that $l_\tau(\gamma) \leq 2\epsilon$. We know that when ϵ small enough, $\gamma \subset B_1$ since the Ricci metric and the Euclidean metric are equivalent on B_1. This implies that the Euclidean length of γ, denoted by $\widetilde{l}(\gamma) \leq c\epsilon$ for some constant c only depending on the comparison constants of the Ricci metric and the Euclidean metric on B_1. It is clear that γ bounds a minimal disk $\widetilde{\Sigma}$ with respect to the Euclidean metric. By the isoperimetric inequality, we know that the Euclidean area $A_E(\widetilde{\Sigma})$ satisfies

$$A_E(\widetilde{\Sigma}) \leq c_1 \widetilde{l}(\gamma)^2 \leq c_1 c^2 \epsilon^2.$$

By using the equivalence of the metrics, we know the area $A_\tau(\widetilde{\Sigma})$ of the surface $\widetilde{\Sigma}$ under the Ricci metric is small if ϵ is small enough. Thus γ bounds a minimal disk Σ with respect to the Ricci metric. By the Gauss-Codazzi equation we know that the curvature R_Σ of the metric on Σ induced from the Ricci metric is bounded above by δ. By using the isoperimetric inequality, we know that

$$A_\tau(\Sigma) \leq c_2 \epsilon^2.$$

However, by the Gauss-Bonnet theorem, since the geodesic γ has at most one vertex p and the outer angle θ at p is at most π, we have

$$\int_\Sigma R_\Sigma \ dv_\tau + \int_\gamma \kappa_\gamma \ ds + \theta = 2\pi\chi(\Sigma) = 2\pi$$

where dv_τ is the induced area form from the Ricci metric. Since γ is a geodesic, we see that the second term in the left hand side of the above formula is 0. Since $R_\Sigma \leq \delta$ and $\theta < \pi$, we have

$$\delta A_\tau(\Sigma) \geq \int_\Sigma R_\Sigma \ dv_\tau = 2\pi - \theta > \pi$$

which implies $A_\tau(\Sigma) \geq \frac{\pi}{\delta}$. By comparing the above two inequalities, we get a contradiction as long as ϵ is small enough. This finishes the proof.

□

4. The Stability of the Logarithmic Cotangent Bundle

In this section we investigate the cohomology classes defined by the currents ω_τ and ω_{KE}. Since both of these Kähler forms have Poincaré growth, it is natural to identify them with the first Chern class of the logarithmic cotangent bundle of $\overline{\mathcal{M}}_g$. This implies this bundle is positive over the compactified moduli space which directly implies that the moduli space is of log general type.

The next step is to show that the restriction of the Kähler-Einstein metric to a subbundle of the logarithmic cotangent bundle will not have growth worse than Poincaré growth. Then we prove that the logarithmic cotangent bundle $\overline{E}$ over $\overline{\mathcal{M}}_g$ is stable with respect to the first Chern class of this bundle.

More precisely we have the following theorem:

Theorem 4.1. *The first Chern class of $\overline{E}$ is positive and $\overline{E}$ is Mumford stable with respect to $c_1(\overline{E})$.*

We first setup our notation. On $\mathcal{M}_g$, let g_p, τ, $\widetilde{\tau}$, g_{WP} and g_{KE} be the asymptotic Poincaré metric, the Ricci metric, the perturbed Ricci metric, the Weil-Petersson metric and the Kähler-Einstein metric respectively. Let ω_p, ω_τ and ω_{KE} be the corresponding Kähler forms of these metrics. Let $Ric(\omega_\tau)$ be the Ricci form of the Ricci metric.

Let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ be the compactification divisor. In order to prove the stability, we need to control the growth of these Kähler forms near D. We fix a cover of $\mathcal{M}_g$ by local charts.

For each point $y \in D$, we can pick local pinching coordinate charts $U_y \subset \widetilde{U}_y$ centered at y with $U_y = (\Delta^*_{\delta_y})^{m_y} \times \Delta^{n-m_y}_{\delta_y}$ and $\widetilde{U}_y = (\Delta^*_{2\delta_y})^{m_y} \times \Delta^{n-m_y}_{2\delta_y}$ such that the estimates in Corollary 4.1, Corollary 4.2, Theorem 4.4 and Theorem 5.2 of [9] hold on $\widetilde{U}_y$. Here Δ_{δ_y} is the disk of radius $\delta_y > 0$ and $\Delta^*_{\delta_y}$ is the punctured disk of radius δ_y and m_y is the codimension of the point y and $n = 3g-3$ is the complex dimension of $\mathcal{M}_g$.

Since D is compact, we can find p such charts $U_1 = U_{y_1}, \cdots, U_p = U_{y_p}$ such that there is a neighborhood V_0 of D with $D \subset V_0 \subset \overline{V_0} \subset \cup_{i=1}^p U_i$. Now we choose coordinate charts $V_1, \cdots, V_q$ such that the estimate of Theorem 5.2 of [9] hold and

(1) $\mathcal{M}_g \subset \left(\cup_{j=1}^p U_j\right) \cup \left(\cup_{j=1}^q V_j\right)$;
(2) $\left(\cup_{j=1}^q V_j\right) \cap \overline{V_0} = \emptyset$.

Let $\psi_1, \cdots, \psi_{p+q}$ be a partition of unity subordinate to the cover $U_1, \cdots, U_p, V_1, \cdots, V_q$ such that $supp(\psi_i) \subset U_i$ for $1 \le i \le p$ and $supp(\psi_i) \subset V_{i-p}$ for $p+1 \le i \le p+q$.

Let $\alpha_i = m_{y_i}$ and let $t^i_1, \cdots, t^i_{\alpha_i}, s^i_{\alpha_i+1}, \cdots, s^i_n$ be the pinching coordinates on U_i where $t^i_1, \cdots, t^i_{\alpha_i}$ represent the degeneration directions.

To prove the theorem, we need a special cut-off function. Such function was used in [10]. We include a short proof here since we need to use the construction later.

Lemma 4.1. *For any small $\epsilon > 0$ there is a smooth function ρ_ϵ such that*

(1) $0 \le \rho_\epsilon \le 1$;
(2) *For any open neighborhood V of D in $\overline{\mathcal{M}}_g$, there is a $\epsilon > 0$ such that $supp(1-\rho_\epsilon) \subset V$;*
(3) *For each $\epsilon > 0$, there is a neighborhood W of D such that $\rho_\epsilon \mid_W \equiv 0$;*
(4) $\rho_{\epsilon'} \ge \rho_\epsilon$ *if* $\epsilon' \le \epsilon$;
(5) *There is a constant C which is independent of ϵ such that*

$$-C\omega_p \le \sqrt{-1}\partial\overline{\partial}\rho_\epsilon \le C\omega_p.$$

Proof. We fix a smooth function $\varphi \in C^\infty(\mathbb{R})$ with $0 \le \varphi \le 1$ such that

$$\varphi(x) = \begin{cases} 0 & x \ge 1; \\ 1 & x \le 0. \end{cases}$$

Now let

$$\varphi_\epsilon(z) = \varphi\left(\frac{\left(\log\frac{1}{|z|}\right)^{-1} - \epsilon}{\epsilon}\right).$$

For $1 \leq i \leq p$ and $\epsilon > 0$ small, we let

$$\varphi_\epsilon^i(t_1^i, \cdots, s_n^i) = \prod_{j=1}^{\alpha_i}(1 - \varphi_\epsilon(t_j^i)).$$

The cut-off function is defined by

$$\rho_\epsilon = 1 - \sum_{i=1}^{p} \psi_i \varphi_\epsilon^i.$$

It is easy to check that ρ_ϵ satisfy all the conditions.

□

Now we discuss the logarithmic cotangent bundle. Let $U_1, \cdots, V_q$ be the cover of $\mathcal{M}_g$ as above. For each $1 \leq i \leq p$, let $W_i = \Delta_{\delta_{y_i}}^{m_{y_i}} \times \Delta_{\delta_{y_i}}^{n-m_{y_i}}$. Then $W_1, \cdots, W_p, V_1, \cdots, V_q$ is a cover of $\overline{\mathcal{M}}_g$. On each U_i, a local holomorphic frame of the holomorphic cotangent bundle $T^*\mathcal{M}_g$ is $dt_1^i, \cdots, dt_{\alpha_i}^i, ds_{\alpha_i+1}^i, \cdots, ds_n^i$. Let

$$e_j^i = \begin{cases} \frac{dt_j^i}{t_j^i} & j \leq \alpha_i; \\ ds_j^i & j \geq \alpha_i + 1. \end{cases} \tag{4.1}$$

The logarithmic cotangent bundle $\overline{E}$ is the extension of $T^*\mathcal{M}_g$ to $\overline{\mathcal{M}}_g$ such that on each U_i, $e_1^i, \cdots, e_n^i$ is a local holomorphic frame of $\overline{E}$. It is very easy to check this fact by writing down the transition maps. In the following, we will use $g_{_{WP}}^*$, τ^* and $g_{_{KE}}^*$ ro represent the metrics on $\overline{E}$ induced by the Weil-Petersson metric, the Ricci metric and the Kähler-Einstein metric respectively.

To discuss the stability of $\overline{E}$, we need to fix a Kähler class on $\overline{\mathcal{M}}_g$. It is natural to use the first Chern class of $\overline{E}$. we denote this class by Φ. We first identify the current represented by the Kähler form $\omega_{_{KE}}$ with Φ.

Lemma 4.2. *The currents ω_τ and $\omega_{_{KE}}$ are positive closed currents. Furthermore,*

$$[\omega_\tau] = [\omega_{_{KE}}] = c_1(\overline{E}).$$

Proof. It is clear that ω_τ and $\omega_{_{KE}}$ are positive currents. Let φ be an arbitrary smooth $(2n-3)$-form on $\overline{\mathcal{M}}_g$. To show that $\omega_{_{KE}}$ is closed, we only need to show

$$\int_{\overline{\mathcal{M}}_g} \omega_{_{KE}} \wedge d\varphi = 0. \tag{4.2}$$

We first check

$$\int_{\overline{\mathcal{M}}_g} |\omega_{_{KE}} \wedge d\varphi| = \int_{\mathcal{M}_g} |\omega_{_{KE}} \wedge d\varphi| < \infty. \tag{4.3}$$

To simplify the notations, on each U_i, we let $t_j^i = s_j^i$ for $\alpha_i + 1 \leq j \leq n$. On each U_i we assume

$$d\varphi = \sum_{\alpha,\beta} a_{\alpha\overline{\beta}}^i dt_1^i \wedge d\overline{t_1^i} \cdots \wedge \widehat{dt_\alpha^i} \wedge d\overline{t_\alpha^i} \cdots \wedge dt_\beta^i \wedge \widehat{d\overline{t_\beta^i}} \cdots \wedge dt_n^i \wedge d\overline{t_n^i}$$

where $a_{\alpha\overline{\beta}}^i$ are bounded smooth functions on U_i. We denote $dt_1^i \wedge d\overline{t_1^i} \cdots \wedge dt_n^i \wedge d\overline{t_n^i}$ by $dt^i \wedge d\overline{t^i}$. By [9] we know that the Kähler-Einstein metric is equivalent to the Ricci metric and the asymptotic Poincaré metric. By using Corollary 4.2 of [9], we know that, restricted to each U_i, there is a constant C depending on φ such that

$$
\begin{aligned}
|\omega_{KE}\wedge d\varphi| &\le \left(\frac{\sqrt{-1}}{2}\right)^n \sum_{\alpha,\beta}\left|(g_{KE})_{\alpha\overline{\beta}}\, a^i_{\alpha\overline{\beta}}\right| dt^i\wedge d\overline{t^i}\\
&\le \left(\frac{\sqrt{-1}}{2}\right)^n C\left(\sum_{j=1}^{\alpha_i}\frac{1}{\left|t^i_j\right|^2\left(\log\left|t^i_j\right|\right)^2}+1\right) dt^i\wedge d\overline{t^i}
\end{aligned}
$$

since $a^i_{\alpha\overline{\beta}}$ is bounded on U_i. This implies

$$
\int_{U_i}\psi_i|\omega_{KE}\wedge d\varphi| \le \int_{U_i}\psi_i\left(\frac{\sqrt{-1}}{2}\right)^n C\left(\sum_{j=1}^{\alpha_i}\frac{1}{\left|t^i_j\right|^2\left(\log\left|t^i_j\right|\right)^2}+1\right) dt^i\wedge d\overline{t^i}<\infty.
$$

So we have

$$
\int_{\mathcal{M}_g}|\omega_{KE}\wedge d\varphi| \le \sum_{j=1}^{q}\int_{V_j}\psi_{j+p}\,|\omega_{KE}\wedge d\varphi| + \sum_{j=1}^{p}\int_{U_j}\psi_j\,|\omega_{KE}\wedge d\varphi|<\infty.
$$

Let ρ_ϵ be the cut-off function constructed above. By the dominating convergence theorem, we have

$$
\int_{\mathcal{M}_g}\omega_{KE}\wedge d\varphi = \lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon\omega_{KE}\wedge d\varphi. \tag{4.4}
$$

Let h be an Hermitian metric on $\overline{E}$. Let $Ric(h)=-\partial\overline{\partial}\log\det(h)$ be its Ricci form. Clearly,

$$
[Ric(h)] = c_1(\overline{E})
$$

and

$$
\lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon Ric(h)\wedge d\varphi = \int_{\mathcal{M}_g}Ric(h)\wedge d\varphi = \int_{\overline{\mathcal{M}}_g}Ric(h)\wedge d\varphi = -\int_{\overline{\mathcal{M}}_g}d(Ric(h))\wedge\varphi = 0.
$$

Since $\omega_{KE}=-\partial\overline{\partial}\log\det(g^*_{KE})$, we have

$$
\begin{aligned}
\int_{\mathcal{M}_g}\omega_{KE}\wedge d\varphi &= \lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon\omega_{KE}\wedge d\varphi = \lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon\omega_{KE}\wedge d\varphi - \lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon Ric(h)\wedge d\varphi\\
&= \lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\rho_\epsilon\partial\overline{\partial}\log\left(\frac{\det(h)}{\det(g^*_{KE})}\right)\wedge d\varphi\\
&= -\lim_{\epsilon\to 0}\int_{\mathcal{M}_g}\log\left(\frac{\det(g^*_{KE})}{\det(h)}\right)\partial\overline{\partial}\rho_\epsilon\wedge d\varphi.
\end{aligned} \tag{4.5}
$$

By using the frame in (4.1), by Theorem 1.4 and Corollary 4.2 of [9] we know that there are positive constants C_1 and C_2 which may depend on φ such that, on each U_i,

$$
C_1 \le \frac{\det(g^*_{KE})}{\det(h)} \le C_2\prod_{j=1}^{\alpha_i}\left(\log\left|t^i_j\right|\right)^2
$$

which implies that there is a constant C_3 such that

$$
\left|\log\left(\frac{\det(g^*_{KE})}{\det(h)}\right)\right| \le C_3 + 2\sum_{j=1}^{\alpha_i}\log\log\frac{1}{\left|t^i_j\right|}. \tag{4.6}
$$

Now by Lemma 4.1 we can pick ϵ_0 small enough such that for any $0 < \epsilon < \epsilon_0$, $supp(1-\rho_\epsilon) \subset V_0$. We also know that $supp(\partial\overline{\partial}\rho_\epsilon) \subset supp(1-\rho_\epsilon)$. Again, by Lemma 4.1, since

$$-C\omega_p \leq \partial\overline{\partial}\rho_\epsilon \leq C\omega_p$$

we know that there is a constant C_4 which depends on φ such that

$$\left|\partial\overline{\partial}\rho_\epsilon \wedge d\varphi\right| \leq \left(\frac{\sqrt{-1}}{2}\right)^n C_4 \left(\sum_{j=1}^{\alpha_i} \frac{1}{\left|t_j^i\right|^2 \left(\log\left|t_j^i\right|\right)^2} + 1\right) dt^i \wedge d\overline{t^i}. \tag{4.7}$$

By combining (4.6) and (4.7), and a simple computation we can show that

$$\int_{U_i} \left|\log\left(\frac{\det(g^*_{KE})}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge d\varphi\right| < \infty. \tag{4.8}$$

From the above argument we know that

$$\left|\int_{\mathcal{M}_g} \log\left(\frac{\det(g^*_{KE})}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge d\varphi\right| = \left|\int_{\mathcal{M}_g \cap supp(\partial\overline{\partial}\rho_\epsilon)} \log\left(\frac{\det(g^*_{KE})}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge d\varphi\right|$$

$$\leq \sum_{j=1}^{p} \int_{U_i \cap supp(1-\rho_\epsilon)} \psi_i \left|\log\left(\frac{\det(g^*_{KE})}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge d\varphi\right| \to 0$$

as ϵ goes to 0 because of (4.8) and the fact that the Lebesgue measure of $U_i \cap supp(1-\rho_\epsilon)$ goes to 0. By combining with (4.5) we know that

$$\int_{\mathcal{M}_g} \omega_{KE} \wedge d\varphi = 0$$

which implies ω_{KE} is a closed current. Similarly we can prove that ω_τ is a closed current by the formula

$$\omega_\tau = -\partial\overline{\partial} \log\det(g^*_{WP})$$

and Corollary 4.1 and 4.2 of [9].

Now we prove the second statement of the lemma. Since ω_{KE} is a closed current, to show it represents the first Chern class of $\overline{E}$, we need to prove that for any closed $(n-1,n-1)$-form $\widetilde{\varphi}$ on $\overline{\mathcal{M}}_g$,

$$\int_{\mathcal{M}_g} \omega_{KE} \wedge \widetilde{\varphi} = \int_{\mathcal{M}_g} c_1(\overline{E}) \wedge \widetilde{\varphi}. \tag{4.9}$$

However, this can be easily proved by using the above argument where we replace $d\varphi$ by $\widetilde{\varphi}$. The same argument works for ω_τ. This finishes the proof.

□

Now we compute the degree of $\overline{E}$. In the following, by degree of a bundle over $\overline{\mathcal{M}}_g$ we always mean the Φ-degree.

Lemma 4.3. *The degree of $\overline{E}$ is given by $\int_{\mathcal{M}_g} \omega^n_{KE}$.*

Proof. Since the degree of $\overline{E}$ is given by

$$\deg(\overline{E}) = \int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega^{n-1}_{KE}$$

we need to show that

$$\int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega^{n-1}_{KE} = \int_{\mathcal{M}_g} \omega^n_{KE}. \tag{4.10}$$

By the property of the asymptotic Poincaré metric, we know that

$$\int_{\mathcal{M}_g} \omega_p^n < \infty. \tag{4.11}$$

Since the Kähler-Einstein metric is equivalent to the asymptotic Poincaré metric, we know that $\int_{\mathcal{M}_g} \omega_{KE}^n < \infty$. Also, since $c_1(\overline{E})$ is a closed $(1,1)$-form on $\overline{\mathcal{M}}_g$ which is compact, we know that any representative of $c_1(\overline{E})$ is bounded on $\overline{\mathcal{M}}_g$. This implies there is a constant C_5 such that $-C_5\omega_p \le c_1(\overline{E}) \le C_5\omega_p$. This implies that $\left|\int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1}\right| < \infty$. By using the notations as in the above lemma we have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_g} c_1(\overline{E}) \wedge \omega_{KE}^{n-1} - \int_{\mathcal{M}_g} \omega_{KE}^n = \int_{\mathcal{M}_g} (c_1(\overline{E}) - \omega_{KE}) \wedge \omega_{KE}^{n-1} \\
=&\lim_{\epsilon\to 0} \int_{\mathcal{M}_g} \rho_\epsilon (c_1(\overline{E}) - \omega_{KE}) \wedge \omega_{KE}^{n-1} = \lim_{\epsilon\to 0} \int_{\mathcal{M}_g} \rho_\epsilon \partial\overline{\partial} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \wedge \omega_{KE}^{n-1} \\
=&\lim_{\epsilon\to 0} \int_{\mathcal{M}_g} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge \omega_{KE}^{n-1} \\
=&\lim_{\epsilon\to 0} \int_{\mathcal{M}_g \cap supp(\partial\overline{\partial}\rho_\epsilon)} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge \omega_{KE}^{n-1}.
\end{aligned} \tag{4.12}$$

Now we show that

$$\int_{\mathcal{M}_g} \left|\log \frac{\det(g_{KE}^*)}{\det(h)}\right| \omega_p^n < \infty. \tag{4.13}$$

Since

$$\int_{\mathcal{M}_g} \left|\log \frac{\det(g_{KE}^*)}{\det(h)}\right| \omega_p^n = \sum_{i=1}^p \int_{U_i} \psi_i \left|\log \frac{\det(g_{KE}^*)}{\det(h)}\right| \omega_p^n + \sum_{j=1}^q \int_{V_j} \psi_{p+j} \left|\log \frac{(g_{KE}^*)}{\det(h)}\right| \omega_p^n$$

and V_j lies in the compact set $\mathcal{M}_g \setminus V_0$ and $0 \le \varphi_i \le 1$, we only need to show that

$$\int_{U_i} \left|\log \frac{(g_{KE}^*)}{\det(h)}\right| \omega_p^n < \infty. \tag{4.14}$$

We know that there is a constant C_i such that, on U_i,

$$\omega_p^n \le C_i \prod_{j=1}^{\alpha_i} \frac{1}{|t_j^i|^2 (\log|t_j^i|)^2}.$$

Formula (4.14) follows from the above formula, inequality (4.6) and a simple computation.

Now we pick ϵ small such that $supp(\partial\overline{\partial}\rho_\epsilon) \subset supp(1-\rho_\epsilon) \subset V_0$. By Lemma 4.1 we know that there is a constant C such that

$$-C\omega_p \le \partial\overline{\partial}\rho_\epsilon \le C\omega_p$$

and

$$0 \le \omega_{KE} \le C\omega_p.$$

So we have

$$\begin{aligned}
&\left|\int_{\mathcal{M}_g \cap supp(\partial\overline{\partial}\rho_\epsilon)} \log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right) \partial\overline{\partial}\rho_\epsilon \wedge \omega_{KE}^{n-1}\right| \\
\le& C^n \sum_{i=1}^p \int_{U_i \cap supp(1-\rho_\epsilon)} \left|\log\left(\frac{\det(g_{KE}^*)}{\det(h)}\right)\right| \omega_p^n \to 0
\end{aligned}$$

as ϵ goes to 0 because of inequality (4.14) and the fact that the Lebesgue measure of

$$U_i \cap supp(1-\rho_\epsilon)$$

goes to 0. By combining with formula (4.12) we have proved this lemma.

□

Now we define the pointwise version of the degree. Let $F \subset \overline{E}$ be a holomorphic subbundle of rank $k \leq n$. Let $g^*_{KE} \mid_F$ and $h \mid_F$ be the restriction to F of the metrics induced by the Kähler-Einstein metric and the metric h. Let

$$d(F) = -\partial\overline{\partial} \log \det(g^*_{KE} \mid_F) \wedge \omega^{n-1}_{KE}. \tag{4.15}$$

The following result is well-known. Please see [5] for details.

Lemma 4.4. *For any holomorphic subbundle F of $\overline{E}$ with rank k, we have*

$$\frac{d(F)}{k} \leq \frac{d(\overline{E})}{n}. \tag{4.16}$$

Now we prove the main theorem.

Proof. Let F be a holomorphic subbundle of $\overline{E}$ of rank k. We first check that $\int_{\mathcal{M}_g} d(F)$ is finite and equal to the degree of F. To prove that $\int_{\mathcal{M}_g} d(F)$ is finite, we need to show that $-\partial\overline{\partial}\log\det(g^*_{KE} \mid_F)$ has Poincaré growth. This involves the estimate of the derivatives of the Kähler-Einstein metric up to second order. Our method is to use Lemma 4.4 together with the monotone convergence theorem and integration by parts to reduce the C^2 estimates of the Kähler-Einstein metric to C^0 estimates.

By Lemma 4.1 we know that ρ_ϵ is monotonically increasing when ϵ is monotonically decreasing. Also by Lemma 4.4 we know that

$$\frac{k}{n}\omega^n_{KE} - d(F) = \frac{k}{n} d(\overline{E}) - d(F) \geq 0.$$

By the monotone convergence theorem we have

$$\lim_{\epsilon \searrow 0} \int_{\mathcal{M}_g} \rho_\epsilon \left(\frac{k}{n}\omega^n_{KE} - d(F) \right) = \int_{\mathcal{M}_g} \left(\frac{k}{n}\omega^n_{KE} - d(F) \right). \tag{4.17}$$

By Lemma 4.3 we have

$$\begin{aligned} \frac{k}{n}\deg(\overline{E}) - \deg(F) &= \int_{\mathcal{M}_g} \left(\frac{k}{n}\omega^n_{KE} - Ric(h \mid_F) \wedge \omega^{n-1}_{KE} \right) \\ &= \lim_{\epsilon \to 0} \int_{\mathcal{M}_g} \rho_\epsilon \left(\frac{k}{n}\omega^n_{KE} - Ric(h \mid_F) \wedge \omega^{n-1}_{KE} \right). \end{aligned} \tag{4.18}$$

However,

$$\begin{aligned} &\int_{\mathcal{M}_g} \rho_\epsilon \left(\frac{k}{n}\omega^n_{KE} - d(F) \right) - \int_{\mathcal{M}_g} \rho_\epsilon \left(\frac{k}{n}\omega^n_{KE} - Ric(h \mid_F) \wedge \omega^{n-1}_{KE} \right) \\ =& \int_{\mathcal{M}_g} \rho_\epsilon \left(Ric(h \mid_F) \wedge \omega^{n-1}_{KE} - d(F) \right) = \int_{\mathcal{M}_g} \rho_\epsilon \left(Ric(h \mid_F) + \partial\overline{\partial}\log\det\left(g^*_{KE} \mid_F\right)\right) \wedge \omega^{n-1}_{KE} \\ =& \int_{\mathcal{M}_g} \rho_\epsilon \partial\overline{\partial} \log \left(\frac{\det\left(g^*_{KE} \mid_F\right)}{\det\left(h \mid_F\right)} \right) \wedge \omega^{n-1}_{KE} = \int_{\mathcal{M}_g} \log \left(\frac{\det\left(g^*_{KE} \mid_F\right)}{\det\left(h \mid_F\right)} \right) \partial\overline{\partial}\rho_\epsilon \wedge \omega^{n-1}_{KE} \\ =& \int_{\mathcal{M}_g \cap supp(1-\rho_\epsilon)} \log \left(\frac{\det\left(g^*_{KE} \mid_F\right)}{\det\left(h \mid_F\right)} \right) \partial\overline{\partial}\rho_\epsilon \wedge \omega^{n-1}_{KE}. \end{aligned} \tag{4.19}$$

Now we show that

$$\lim_{\epsilon\to 0}\int_{\mathcal{M}_g\cap supp(1-\rho_\epsilon)}\log\left(\frac{\det\left(g^*_{KE}\mid_F\right)}{\det\left(h\mid_F\right)}\right)\partial\overline{\partial}\rho_\epsilon\wedge\omega_{KE}^{n-1}=0. \tag{4.20}$$

By the proof of Lemma 4.3, to prove formula (4.20) we only need to show that

$$\int_{\mathcal{M}_g}\left|\log\left(\frac{\det\left(g^*_{KE}\mid_F\right)}{\det\left(h\mid_F\right)}\right)\right|\omega_p^n<\infty \tag{4.21}$$

which is reduced to show that

$$\int_{U_i}\left|\log\left(\frac{\det\left(g^*_{KE}\mid_F\right)}{\det\left(h\mid_F\right)}\right)\right|\omega_p^n<\infty. \tag{4.22}$$

Since the Ricci metric is equivalent to the Kähler-Einstein metric, we know that $\frac{\det\left(g^*_{KE}|_F\right)}{\det(\tau^*|_F)}$ is bounded from above and below by positive constants.

We fix a U_i. Let $\{f_1,\cdots,f_k\}$ be a local holomorphic frame of F. We know that there exists a $k\times n$ matrix $B=(b_{\alpha\beta})$ whose entries are holomorphic functions on U_i such that the rank of B is k and $f_\alpha=\sum_\beta b_{\alpha\beta}e^i_\beta$ where e^i_β is defined in (4.1). Now we have

$$\frac{\det\left(\tau^*\mid_F\right)}{\det\left(h\mid_F\right)}=\frac{\det\left(B\left(\tau^*_{i\overline{j}}\right)\overline{B}^T\right)}{\det\left(B\left(h_{i\overline{j}}\right)\overline{B}^T\right)}.$$

Now we need the following linear algebraic lemma:

Lemma 4.5. *For any positive Hermitian $n\times n$ matrix A, we denote its eigenvalues by $\lambda_1,\cdots,\lambda_n$ where $\lambda_i=\lambda_i(A)$ such that $\lambda_1(A)\geq\cdots\geq\lambda_n(A)>0$. Let A_1 and A_2 be two positive Hermitian $n\times n$ matrices. Let B be an $k\times n$ matrix with $k\leq n$ such that the rank of B is k. Then there are positive constants c_1 and c_2 only depending on n,k such that*

$$c_1\frac{\lambda_{n-k+1}(A_1)\cdots\lambda_n(A_1)}{\lambda_1(A_2)\cdots\lambda_k(A_2)}\leq\frac{\det\left(BA_1\overline{B}^T\right)}{\det\left(BA_2\overline{B}^T\right)}\leq c_2\frac{\lambda_1(A_1)\cdots\lambda_k(A_1)}{\lambda_{n-k+1}(A_2)\cdots\lambda_n(A_3)}.$$

We briefly show the proof here.

Proof. We fix A_1 and A_2 and let

$$Q(B)=\frac{\det\left(BA_1\overline{B}^T\right)}{\det\left(BA_2\overline{B}^T\right)}.$$

Let B_i be the $k\times n$ matrix obtained by multiplying the i-th row of B by a non-zero constant c and leave other rows invariant and let B_{ij} be the $k\times n$ matrix obtained by adding a constant multiple of the j-th row to the i-th row and leave other rows invariant. It is easy to check that $Q(B_i)=Q(B_{ij})=Q(B)$.

Thus we can assume that the row vectors of B form an orthonormal set of $\mathbb{C}^n$. With this assumption, it is easy to see that there are positive constants c_3 and c_4 only depending on n,k such that

$$c_3\lambda_{n-k+1}(A_i)\cdots\lambda_n(A_i)\leq\det\left(BA_i\overline{B}^T\right)\leq c_4\lambda_1(A_i)\cdots\lambda_k(A_i)$$

for $i=1,2$. The lemma follows directly.

□

Now we go back to the proof of the theorem. By using Theorem 1.4 and corollary 4.2 of [9], we know that, under the frame (4.1),

(1) $\tau^*_{i\bar{i}} = u_i^{-2}(1+O(u_0))$ if $i \le m$;
(2) $\tau^*_{i\bar{j}} = O(1)$ if $i,j \le m$ with $i \ne j$ or $i \le m < j$ or $j \le m < i$;
(3) $\tau^*_{i\bar{j}} = \tau^{i\bar{j}}$ if $i,j \ge m+1$;
(4) On U_i, the submatrix $\left(\tau^{i\bar{j}}\right)_{i,j\ge m+1}$ is bounded from above and below by positive constants multiple of the identity matrix where the constants depend on U_i.

It is clear that, on U_i the eigenvalues of matric matrix of h with respect to the frame (4.1) are bounded from above and below by positive constants which depend on U_i and the choice of the metric h.

By analyzing the eigenvalues of the matrix (τ^*) and by using Lemma 4.5, a simple computation shows that there are positive constants C_6 and C_7 which depend on F and U_i such that

$$\left|\log\left(\frac{\det\left(g^*_{KE}\mid_F\right)}{\det\left(h\mid_F\right)}\right)\right| \le C_6 + C_7\sum_{j=1}^{\alpha_i}\log\log\frac{1}{\left|t^i_j\right|}. \tag{4.23}$$

Now by using a similar method to the proof of Lemma 4.3 we know that formula (4.22) and (4.21) hold which imply formula (4.20) holds. Combining (4.20) and (4.19) we have

$$\lim_{\epsilon\searrow 0}\left(\int_{\mathcal{M}_g}\rho_\epsilon\left(\frac{k}{n}\omega^n_{KE} - d(F)\right) - \int_{\mathcal{M}_g}\rho_\epsilon\left(\frac{k}{n}\omega^n_{KE} - Ric(h\mid_F)\wedge\omega^{n-1}_{KE}\right)\right) = 0. \tag{4.24}$$

By combining (4.24), (4.18) and (4.17) we have

$$\int_{\mathcal{M}_g}\left(\frac{k}{n}\omega^n_{KE} - d(F)\right) = \frac{k}{n}\deg(\overline{E}) - \deg(F). \tag{4.25}$$

By Lemma 4.3 we know that $\int_{\mathcal{M}_g}\omega^n_{KE} = \deg(\overline{E})$. From formula (4.25) we know that $\int_{\mathcal{M}_g} d(F)$ is finite and

$$\deg(F) = \int_{\mathcal{M}_g} d(F). \tag{4.26}$$

Now by Lemma 4.4 we have

$$\frac{\deg(F)}{k} - \frac{\deg(\overline{E})}{n} = \int_{\mathcal{M}_g}\left(\frac{d(F)}{k} - \frac{d(\overline{E})}{n}\right) \le 0$$

which implies

$$\frac{\deg(F)}{k} \le \frac{\deg(\overline{E})}{n}.$$

This proves that the bundle $\overline{E}$ is semi-stable in the sense of Mumford.

To prove the strict stability of the logarithm cotangent bundle, we need to show that this bundle cannot split. The following result about the moduli group Mod_g and its proof is due to F. Luo [11].

Proposition 4.1. *Let Mod_g be the moduli group of closed Riemann surfaces of genus g with $g \ge 2$. Then any finite index subgroup of Mod_g is not isomorphic to a product of groups.*

The proof of the above proposition is topological. For completeness, we will include Luo's proof at the end of this section.

Now we go back to the proof of stability. If $\overline{E}$ is not stable, then it must split into a direct sum of holomorphic subbundles $\overline{E} = \oplus_{i=1}^k E_i$ with $k \ge 2$. Moreover, when restricted to the moduli space, both the connection and the Kähler-Einstein metric split. It is well known that there is a finite cover $\widetilde{M}_g$ of $\mathcal{M}_g$ which is smooth. By the decomposition theorem of de Rham, the Teichmüller space, as the universal covering space of $\widetilde{M}_g$ must split isomorphically as a product of manifolds. Furthermore, the fundamental group of $\widetilde{M}_g$ is isomorphic to a product of groups. However, $\pi_1(\widetilde{M}_g)$ is a finite index subgroup of the mapping class group. By the above proposition, this is impossible. So we have proved the stability.

□

We remark that the positivity of $c_1(\overline{E})$ implies that the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ is of logarithmic general type for $g > 1$.

In the end of this section we give a proof by F. Luo of Proposition 4.1.

Proof of Proposition 4.1. The proof uses Thurston's classification of elements in the mapping class group [16] and the solution of the Nielsen realization problem [4]. In the following we will use the words "simple loops" and "subsurfaces" to denote the isotopy classes of simple loops and subsurfaces. We fix a surface X.

Suppose there is a subgroup G of Mod_g with finite index such that there are two nontrivial subgroups A and B of G so that $G = A \times B$. We will derive a contradiction. We need to following lemma.

Lemma 4.6. *Let A, B and G be as above. Then*

(1) *There are elements of infinite order in both A and B;*
(2) *There are no elements in A or B which is pseudo-Anosov.*

Proof. If the first claim is not true, then we can assume A consists of torsions only. We claim that, in this case, A is a finite group.

Actually let $\pi : \mathrm{Mod}_g \to \mathrm{Aut}(H_1(X))$ be the natural homomorphism. By the virtual of Theorem V.3.1 of [3], we know that the kernel of π contains no torsion elements. Thus $\pi(A)$ is isomorphic to A. Now $\pi(A)$ is a torsion subgroup of the general linear group $\mathrm{GL}(n, \mathbf{R})$. By the well known solution of the Burnside problem for the linear groups, we see that $\pi(A)$ must be finite and so is A.

For the finite group A, by the solution of the Nielsen realization problem of Kerckhoff [4], we know that there is a point d in the Teichmüller space of X fixed by all elements in A. Let $\mathrm{Fix}(A)$ be the set of all points in the Teichmüller space fixed by each element in A. Then we see that $\mathrm{Fix}(A)$ is a non-empty proper subset of the Teichmüller space. Now for each $a \in A$, $b \in B$ and $d \in \mathrm{Fix}(A)$, since $ab = ba$, we have $ab(d) = ba(d) = b(d)$. This implies $\mathrm{Fix}(A)$ is invariant under the action of B on the Teichmüller space. Thus we see that the finite index subgroup $G = A \times B$ acts on the Teichmüller space leaving $\mathrm{Fix}(A)$ invariant. This contradicts the finite index property of G since $\mathrm{Fix}(A)$ is the Teichmüller space of the orbifold X/A. This proved the first claim.

Now we check the second claim. If it is not true, then we can assume that there is an $a \in A$ which is pseudo-Anosov. Now we consider the action of the mapping class group on the space of all measured laminations in the surface. By Thurston's theory, there are exactly two measured laminations m, m' fixed by a. Now for all $b \in B$, due to $ab = ba$, we see that b leaves $\{m, m'\}$ invariant. A result of McCarthy [13] shows that the stablizers of $\{m, m'\}$ in the mapping class group is virtually cyclic. Thus we see that each element $b \in B$ has some power b^n with $n \neq 0$ which is equal to a^k with $k \neq 0$. By the first claim we know that B contains elements of infinite order. This implies that some power a^k with $k \neq 0$ is in B which is a contradiction. This proved the second claim.

□

Now we go back to the proof of the proposition. By the above lemma, we can take $a \in A$ and $b \in B$, both are infinite order and none of them is pseudo-Anosov. Thus by replacing a and b by a high power a^n and b^n with $n > 0$, we may assume that for a there is a set of disjoint simple loops $c_1, \cdots, c_k$ in X so that

(1) Each component of $X \setminus \cup_{i=1}^k c_i$ is invariant under a;

(2) The restriction of a to each component of $X \setminus \cup_{i=1}^{k} c_i$ is the identity map or a pseudo-Anosov map.

Let $N(c_i)$ be a regular neighborhood of c_i in X and let $F(a)$ be the subsurface which is the union of all $N(c_i)$'s with those components of $X \setminus \cup_{i=1}^{k} c_i$ on which a is the identity map. By Thurston's classification, the subsurface $F(a)$ has the property that if c is a curve system invariant under a, then c is in $F(a)$. We can construct $F(b)$ in a similar way.

Now since $ab = ba$ we know that if a simple loop c is invariant under b, then $a(c)$ is also invariant under b because $ba(c) = ab(c) = a(c)$. Thus for all simple loops c in $F(b)$, $a(c)$ is still in $F(b)$. This implies that $a(F(b)) = F(b)$. By using the property of $F(a)$, we see $F(b) \subset F(a)$. Similar reason implies $F(a) \subset F(b)$. Thus $F(a) = F(b)$.

Now we show that the subsurface $F(a) = F(b)$ is invariant under each element in A and B. We pick $x \in A$. Due to $xb = bx$, we have $x(F(b)) = b(x(F(b)))$ which implies $x(F(b)) \subset F(b)$. Since these two surfaces are homeomorphic, we have $x(F(b)) = F(b)$. Similarly, for all $y \in B$, $y(F(a)) = F(a)$. Thus all elements in $G = A \times B$ leave the subsurface $F(a)$ invariant.

Now there are two cases. In the first case, $F(a)$ is not homeomorphic to X. In the second case, $F(a) = F(b) = X$.

It is clear that the first case $F(a)$ is not homeomorphic to X cannot occur. Otherwise, the finite index subgroup G leaves a proper subsurface $F(a)$ invariant. This contradicts the known properties of the mapping class group. As conclusion of this case, we see that for any element $x \in A$, any power x^n of x cannot contain pseudo-Anosov components in Thurston's classification.

In the second case, $F(a) = F(b) = X$. In this case, some power of a (and b) is a composition of Dehn twists on disjoint simple loops. By the argument in the first case, we see that for any two elements $x \in A$ and $y \in B$, some powers x^n, y^m with $m, n \neq 0$ are either the identity map or compositions of Dehn twists on disjoint simple loops.

let $\mathrm{Fix}(x)$ be the union of the disjoint simple loops so that x is the composition of Dehn twists on these simple loops. We define $\mathrm{Fix}(x)$ to be the empty set if $x^n = id$ for some non-zero integer n. We need the following claim:

Claim 1. *For any $x \in A$ and $y \in B$, the geometric intersection number between Fix(x) and Fix(y) is zero.*

To prove the claim, without loss of generality, we may take $x = a$ and $y = b$. We may assume that a is the composition of Dehn twists on curve system $\mathrm{Fix}(a) = c$ and b is the composition of Dehn twists on curve system $\mathrm{Fix}(b) = d$. Note that in this case, if s is a curve system invariant under a, then the geometric intersection number between s and c is zero. Namely $I(s, c) = 0$. We also say that s is *disjoint* from c. Now since $ab = ba$ and $a(c) = c$, we have that $a(b(c)) = b(c)$. Thus the curve system $b(c)$ is disjoint from c. Since b is a composition of Dehn twists on disjoint simple loops d, this shows that $I(c, d) = 0$. Namely d and c are disjoint curve systems.

Now we finish the proof of the proposition as follows. First of all, by the assumption $\mathrm{Fix}(a)$ and $\mathrm{Fix}(b)$ are both non-empty. Let S_1 (and S_2) be the smallest subsurface of X which contains all curves in $\mathrm{Fix}(x)$ for $x \in A$ (or $x \in B$). By the above claim, we have $S_1 \cap S_2 = \emptyset$. Thus there is an essential simple loop c which is disjoint from both S_1 and S_2. By the construction, $x(c) = c$ and $y(c) = c$ for all $x \in A$ and $y \in B$. Thus we see that for each element $e \in G = A \times B$, there is a power e^n with $n \neq 0$ so that e^n leaves c invariant. This contradicts the fact that $A \times B$ is a finite index subgroup of the mapping class group. This finishes the proof.

□

5. The Bounded Geometry of the Kähler-Einstein Metric

In this section we show that the Kähler-Einstein metric has bounded curvature and the injectivity radius of the Teichmüller space equipped with the Kähler-Einstein metric is bounded from below.

We begin with a metric that is equivalent to the Kähler-Einstein metric and its curvature as well as the covariant derivatives of the curvature are uniformly bounded. We deform the Ricci metric whose curvature is bounded to obtain this metric by using Kähler-Ricci flow. Then we establish the Monge-Amperé equation from this new metric. Our work in [9] implies the new metric is equivalent to the Kähler-Einstein metric which gave us C^2 estimates. Based on this, we do the C^3 and C^4 estimates. This will give us the boundedness of the curvature of the Kähler-Einstein metric. The same method can be used to show that all the covariant derivatives of the Kähler-Einstein metric are bounded.

We slightly change our notations. We will use g to denote the Kähler-Einstein metric and use h to denote an equivalent metric whose curvature and covariant derivatives of curvature are bounded. We use $z_1, \cdots, z_n$ to denote local holomorphic coordinates on the Teichmüller space. The main result of this section is the following theorem:

Theorem 5.1. *Let g be the Kähler-Einstein metric on the Teichmüller space $\mathcal{T}$. Then the curvature of g and all of its covariant derivatives are all bounded.*

Proof. We begin with the Ricci metric τ. We first obtain a new equivalent metric h by deforming τ with the Ricci flow. Consider the following Kähler-Ricci flow:

$$\begin{cases} \frac{\partial g_{i\overline{j}}}{\partial t} = -(R_{i\overline{j}} + g_{i\overline{j}}) \\ g_{i\overline{j}}(0) = \tau_{i\overline{j}} \end{cases} . \tag{5.1}$$

If we let $s = e^t - 1$ and $\widetilde{g} = e^t g$, we have

$$\begin{cases} \frac{\partial \widetilde{g}_{i\overline{j}}}{\partial s} = -\widetilde{R}_{i\overline{j}} \\ \widetilde{g}_{i\overline{j}}(0) = \tau_{i\overline{j}} \end{cases} . \tag{5.2}$$

Since the initial metric has bounded curvature, by the work of Shi [15], the flow (5.2) has short time existence and for small s, the metric $\widetilde{g}(s)$ is equivalent to the initial metric τ. Furthermore, the curvature and its covariant derivatives of $\widetilde{g}(s)$ are bounded. Hence for small t, the metric $g(t)$ is equivalent to the Ricci metric τ and has bounded curvature as well as covariant derivatives of the curvature.

Now we fix a small t and denote the metric $g(t)$ by h. Since the Teichmüller space is contractible, there are smooth functions u and F such that

$$\omega_g = \omega_h + \partial\overline{\partial} u \tag{5.3}$$

and

$$Ric(h) + \omega_h = \partial\overline{\partial} F. \tag{5.4}$$

Since the metrics h and τ are equivalent, we know that h and g are equivalent which implies the tensor $u_{i\overline{j}} dz_i \otimes d\overline{z}_j$ is bounded with respect to either metric. Also, because the curvature and its covariant derivatives of the metric h are bounded, we know that a covariant derivative of F with respect to h is bounded if this derivative is at least order 2 and has at least one holomorphic direction and one anti-holomorphic direction. So we have C^2 estimates.

By the Kähler-Einstein condition of g, we have the Monge-Ampère equation

$$\log\det(h_{i\overline{j}} + u_{i\overline{j}}) - \log\det(h_{i\overline{j}}) = u + F. \tag{5.5}$$

We use Δ, Δ', ∇, ∇', Γ^k_{ij}, $\widetilde{\Gamma}^k_{ij}$, $R_{i\overline{j}k\overline{l}}$, $P_{i\overline{j}k\overline{l}}$, $R_{i\overline{j}}$, $P_{i\overline{j}}$, R and P to denote the Laplacian, gradient, Christoffell symbol, curvature tensor, Ricci curvature and scalar curvature of the metrics h and g respectively. In the following, all covariant derivatives of functions and tensors are taken with respect to the background metric h.

Inspired by Yau's work in [19], we let

$$\mathcal{F} = u + F, \tag{5.6}$$

$$S = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}} \tag{5.7}$$

and

$$V = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{n}kp} u_{;\bar{j}m\bar{l}\bar{q}}. \tag{5.8}$$

To simplify the notation, we define the following quantities:

$$\begin{aligned} A_{ipkm\alpha} =& u_{;i\overline{p}k\overline{m}\alpha} - g^{\gamma\overline{\delta}} u_{;i\overline{\delta}\alpha} u_{;\gamma\overline{p}k\overline{m}} - g^{\gamma\overline{\delta}} u_{;k\overline{\delta}\alpha} u_{;i\overline{p}\gamma\overline{m}} - g^{\gamma\overline{\delta}} u_{;\gamma\overline{p}\alpha} u_{;i\overline{\delta}k\overline{m}} \\ & - g^{\gamma\overline{\delta}} u_{;\gamma\overline{m}\alpha} u_{;i\overline{p}k\overline{\delta}} + g^{\gamma\overline{\delta}} u_{;i\overline{p}\gamma} u_{;\overline{\delta}k\overline{m}\alpha} + g^{\gamma\overline{\delta}} u_{;i\overline{p}\gamma} u_{;k\overline{m}\alpha\overline{\delta}} + 2g^{\gamma\overline{\delta}} u_{;\overline{p}i\overline{\delta}} u_{;k\overline{m}\alpha\gamma}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} B_{ipkm\alpha} =& u_{;\bar{i}p\bar{k}m\alpha} - g^{\gamma\overline{\delta}} u_{;\gamma\bar{i}\alpha} u_{;\overline{\delta}p\bar{k}m} - g^{\gamma\overline{\delta}} u_{;\gamma\bar{k}\alpha} u_{;\bar{i}p\overline{\delta}m} - g^{\gamma\overline{\delta}} u_{;p\overline{\delta}\alpha} u_{;\bar{i}\gamma\bar{k}m} \\ & - g^{\gamma\overline{\delta}} u_{;m\overline{\delta}\alpha} u_{;\bar{i}p\bar{k}\gamma} + g^{\gamma\overline{\delta}} u_{;p\bar{i}\gamma} u_{;\bar{k}m\overline{\delta}\alpha} + g^{\gamma\overline{\delta}} u_{;\bar{i}p\overline{\delta}} u_{;\gamma\bar{k}m\alpha}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} C_{ipkm\alpha} =& u_{;i\overline{m}kp\alpha} - g^{\gamma\overline{\delta}} u_{;i\overline{\delta}\alpha} u_{;\gamma\overline{m}kp} - g^{\gamma\overline{\delta}} u_{;k\overline{\delta}\alpha} u_{;i\overline{m}\gamma p} - g^{\gamma\overline{\delta}} u_{;p\overline{\delta}\alpha} u_{;i\overline{m}k\gamma} \\ & - g^{\gamma\overline{\delta}} u_{;\gamma\overline{m}\alpha} u_{;i\overline{\delta}kp} + g^{\gamma\overline{\delta}} u_{;i\overline{m}\gamma} u_{;k\overline{\delta}p\alpha}, \end{aligned} \tag{5.11}$$

$$\begin{aligned} D_{ipkm\alpha} =& u_{;\bar{i}m\bar{k}\overline{p}\alpha} - g^{\gamma\overline{\delta}} u_{;\gamma\bar{i}\alpha} u_{;\overline{\delta}m\bar{k}\overline{p}} - g^{\gamma\overline{\delta}} u_{;\gamma\bar{k}\alpha} u_{;\bar{i}m\overline{\delta}\overline{p}} - g^{\gamma\overline{\delta}} u_{;\gamma\overline{p}\alpha} u_{;\bar{i}m\bar{k}\overline{\delta}} \\ & - g^{\gamma\overline{\delta}} u_{;m\overline{\delta}\alpha} u_{;\bar{i}\gamma\bar{k}\overline{p}} + g^{\gamma\overline{\delta}} u_{;\bar{i}m\overline{\delta}} u_{;\bar{k}\gamma\overline{p}\alpha}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} W =& g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} u_{;i\bar{q}k\bar{n}\alpha} u_{;\bar{j}p\bar{l}m\overline{\beta}} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} u_{;i\bar{q}k\bar{n}\overline{\beta}} u_{;\bar{j}p\bar{l}m\alpha} \\ & + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} u_{;i\bar{n}kp\alpha} u_{;\bar{j}m\bar{l}\bar{q}\overline{\beta}} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} u_{;i\bar{n}kp\overline{\beta}} u_{;\bar{j}m\bar{l}\bar{q}\alpha} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \widetilde{W} =& g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} \left(A_{iqkn\alpha} \overline{A_{jplm\beta}} + B_{jplm\alpha} \overline{B_{iqkn\beta}} \right) \\ & + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} g^{\alpha\overline{\beta}} \left(C_{ipkn\alpha} \overline{C_{jqlm\beta}} + B_{jqlm\alpha} \overline{B_{ipkn\beta}} \right). \end{aligned} \tag{5.14}$$

Firstly, a simple computation shows that

$$\widetilde{\Gamma}^{\alpha}_{ik} - \Gamma^{\alpha}_{ik} = g^{\alpha\overline{\beta}} u_{;i\overline{\beta}k}. \tag{5.15}$$

Now we compute $P_{i\bar{j}k\bar{l}}$. We first note that

$$u_{;i\bar{j}k\bar{l}} = u_{i\bar{j}k\bar{l}} - u_{\bar{j}p\bar{l}} \Gamma^p_{ik} - u_{i\bar{q}k} \overline{\Gamma^q_{jl}} + u_{p\bar{q}} \Gamma^p_{ik} \overline{\Gamma^q_{jl}} - u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}}.$$

Since $g_{i\bar{j}} = h_{i\bar{j}} + u_{i\bar{j}}$ we have

$$\begin{aligned} P_{i\bar{j}k\bar{l}} =& \partial_k \partial_{\bar{l}} g_{i\bar{j}} - g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{l}} g_{p\bar{j}} = \partial_k \partial_{\bar{l}} h_{i\bar{j}} + u_{i\bar{j}k\bar{l}} - g_{p\bar{q}} \widetilde{\Gamma}^p_{ik} \overline{\widetilde{\Gamma}^q_{jl}} \\ =& \partial_k \partial_{\bar{l}} h_{i\bar{j}} + u_{i\bar{j}k\bar{l}} - g_{p\bar{q}} \left(\Gamma^p_{ik} + g^{p\overline{\beta}} u_{;i\overline{\beta}k} \right) \left(\overline{\Gamma^q_{jl}} + g^{\alpha\bar{q}} u_{;\bar{j}\alpha\bar{l}} \right) \\ =& \left(\partial_k \partial_{\bar{l}} h_{i\bar{j}} - h_{p\bar{q}} \Gamma^p_{ik} \overline{\Gamma^q_{jl}} \right) - u_{p\bar{q}} \Gamma^p_{ik} \overline{\Gamma^q_{jl}} \\ & + \left(u_{i\bar{j}k\bar{l}} - u_{\bar{j}p\bar{l}} \Gamma^p_{ik} - u_{i\bar{q}k} \overline{\Gamma^q_{jl}} + 2u_{p\bar{q}} \Gamma^p_{ik} \overline{\Gamma^q_{jl}} \right) - g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}} \\ =& R_{i\bar{j}k\bar{l}} + u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} + u_{;i\bar{j}k\bar{l}} - g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}}. \end{aligned} \tag{5.16}$$

Since the curvature of the background metric h and the tensor $u_{i\bar{j}} dz_i \otimes d\bar{z}_j$ are both bounded, to prove that the curvature of the metric g is bounded, we only need to show that both S and V are bounded.

We first consider the quantity S. We follow the idea of Yau in [19] and use the following notations:

Definition 5.1. *Let A and B be two functions. We denote*

(1) $A \overset{3}{\simeq} B$ *if* $|A-B| \le C_1\sqrt{S}+C_2$*;*
(2) $A \overset{4}{\simeq} B$ *if* $|A-B| \le C_1\sqrt{V}+C_2$*;*
(3) $A \overset{3}{\cong} B$ *if* $|A-B| \le C_1 S + C_2\sqrt{S}+C_3$*;*
(4) $A \overset{4}{\cong} B$ *if* $|A-B| \le C_1 V + C_2\sqrt{V}+C_3$

where C_1, C_2 and C_3 are universal constants.

Also, by diagonalizing we mean to choose holomorphic coordinates $z_1, \cdots, z_n$ such that

$$h_{i\bar{j}} = \delta_{ij}$$

and

$$u_{i\bar{j}} = \delta_{ij} u_{i\bar{i}}.$$

Now we differentiate the equation (5.5) twice and reorganize the terms. We have

$$g^{i\bar{j}} u_{;i\bar{j}k\bar{l}} = \mathcal{F}_{k\bar{l}} + g^{i\bar{j}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}}. \tag{5.17}$$

By differentiating this equation once more we have

$$\begin{aligned} g^{i\bar{j}} u_{;i\bar{j}k\bar{l}\alpha} =& \mathcal{F}_{;k\bar{l}\alpha} + g^{i\bar{j}} g^{p\bar{q}} \left(u_{;i\bar{q}\alpha} u_{;p\bar{j}k\bar{l}} + u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}\alpha} + u_{;i\bar{q}k\alpha} u_{;\bar{j}p\bar{l}} \right) \\ &- g^{i\bar{j}} g^{p\bar{q}} g^{m\bar{n}} \left(u_{;\bar{j}m\bar{l}} u_{;p\bar{n}\alpha} u_{;i\bar{q}k} + u_{;m\bar{q}k} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} \right). \end{aligned} \tag{5.18}$$

Since

$$\partial_k(\Delta u) = \partial_k \left(h^{i\bar{j}} u_{i\bar{j}} \right) = h^{i\bar{j}} \left(u_{i\bar{j}k} - u_{p\bar{j}} \Gamma^p_{ik} \right) = h^{i\bar{j}} u_{;i\bar{j}k},$$

by diagonalizing and the Schwarz inequality we have

$$\left| \nabla^{'}(\Delta u) \right|^2 = g^{i\bar{j}} \left(h^{k\bar{l}} u_{;k\bar{l}i} \right) \left(h^{p\bar{q}} u_{;\bar{q}p\bar{j}} \right) = \sum_i \frac{1}{1+u_{i\bar{i}}} \left| \sum_k u_{;k\bar{k}i} \right|^2 \le C_1 S \tag{5.19}$$

since the metrics h and g are equivalent and $S = \sum_{i,p,k} \frac{1}{1+u_{i\bar{i}}} \frac{1}{1+u_{p\bar{p}}} \frac{1}{1+u_{k\bar{k}}} |u_{;i\bar{p}k}|^2$. We also have

$$\Delta^{'}(\Delta u) = g^{k\bar{l}} \partial_k \partial_{\bar{l}} \left(h^{i\bar{j}} u_{i\bar{j}} \right) = g^{k\bar{l}} h^{i\bar{j}} u_{;i\bar{j}k\bar{l}} = g^{k\bar{l}} h^{i\bar{j}} \left(u_{;k\bar{l}i\bar{j}} - u_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} + u_{p\bar{l}} h^{p\bar{q}} R_{k\bar{q}i\bar{j}} \right).$$

By using equation (5.17) we have

$$\begin{aligned} \Delta^{'}(\Delta u) =& h^{i\bar{j}} \left(\mathcal{F}_{i\bar{j}} + g^{k\bar{l}} g^{p\bar{q}} u_{;k\bar{q}i} u_{;\bar{l}p\bar{j}} \right) - h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}} + h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} \\ =& \widetilde{S} + \Delta\mathcal{F} - h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}} + h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} \end{aligned} \tag{5.20}$$

where $\widetilde{S} = h^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;k\bar{q}i} u_{;\bar{l}p\bar{j}}$. Since the metrics h and g are equivalent, we know that there is a constant C_2 such that

$$\widetilde{S} \ge C_2 S.$$

Now the term $\left| h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{j}} R_{i\bar{q}k\bar{l}} \right|$ is bounded since h is equivalent to g, the curvature of h is bounded and we have C^2 estimates on u. Similarly, $|\Delta\mathcal{F}|$ is bounded. Finally, since

$$h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} = -g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}} = -g^{k\bar{l}} h^{p\bar{q}} (g_{p\bar{l}} - h_{p\bar{l}}) R_{k\bar{q}} = g^{k\bar{l}} R_{k\bar{q}} - R,$$

we know that $\left| h^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} u_{p\bar{l}} R_{k\bar{q}i\bar{j}} \right|$ is also bounded for similar reasons. By combining the above argument we know that there is a constant C_3 such that

$$\Delta^{'}(\Delta u) \ge C_2 S - C_3. \tag{5.21}$$

Now by the computation in [19], we know there are positive constants C_4, C_5 and C_6 such that

$$\Delta^{'}(S + C_4\Delta u) \geq C_5 S - C_6. \tag{5.22}$$

So for any positive $\lambda > 0$, we can find a positive constant C_7 such that

$$S + C_4\Delta u + C_7 \geq 0 \tag{5.23}$$

and

$$\Delta^{'}(S + C_4\Delta u + C_7) \geq -\lambda(S + C_4\Delta u + C_7) \tag{5.24}$$

since Δu is bounded. We fix λ and let $f = S + C_4\Delta u + C_7$. Now we know that the Ricci curvature of g is -1 and g is equivalent to the Ricci metric τ whose injectivity radius has a lower bound, by the work of Li and Schoen [8] and Li [7] we can find a positive r_0 such that the mean value inequality

$$f(p) \leq C V_p^{-1}(r_0) \int_{B_p(r_0)} f \; dV \tag{5.25}$$

hold for any p in the Teichmüller space. Here $V_p(r_0)$ is the volume of the Kähler-Einstein ball centered at p with radius r_0, $dV = \omega_g^n$ is the volume element of the metric g and C is a constant depending on r_0 and λ but is independent of p. Let $r(z)$ be the function on $B_p(2r_0)$ measuring the g-distance between z and p. We fix a small r_0 and let $\rho = \rho(r)$ be a cutoff function such that $0 \leq \rho \leq 1$, $\rho(r) = 1$ for $r \leq r_0$ and $\rho(r) = 0$ for $r \geq 2r_0$. Since $\Delta^{'}(\Delta u) + C_3 \geq C_2 S \geq 0$, we have

$$\begin{aligned} C_2 \int_{B_p(2r_0)} \rho^2 S \; dV - C_3 V_p(2r_0) \leq & \int_{B_p(2r_0)} \rho^2 \Delta^{'}(\Delta u) \; dV = -2 \int_{B_p(2r_0)} \nabla^{'}\rho \cdot (\rho \nabla^{'}(\Delta u)) \; dV \\ \leq & C_8 \left(\int_{B_p(2r_0)} \left|\nabla^{'}\rho\right|^2 \; dV \right)^{\frac{1}{2}} \left(\int_{B_p(2r_0)} \rho^2 \left|\nabla^{'}(\Delta u)\right|^2 \; dV \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\left|\nabla^{'}\rho\right|$ is bounded and $\left|\nabla^{'}(\Delta u)\right|^2 \leq C_1 S$, we have

$$C_2 \int_{B_p(2r_0)} \rho^2 S \; dV - C_3 V_p(2r_0) \leq C_9 \left(\int_{B_p(2r_0)} \rho^2 S \; dV \right)^{\frac{1}{2}} (V_p(2r_0))^{\frac{1}{2}}$$

which implies

$$\int_{B_p(2r_0)} \rho^2 S \; dV \leq C_{10} V_p(2r_0). \tag{5.26}$$

By using inequalities (5.25)and (5.26) we have

$$\begin{aligned} f(p) \leq & C V_p^{-1}(r_0) \int_{B_p(r_0)} (S + C_4\Delta u + C_7) \; dV \leq C V_p^{-1}(r_0) \int_{B_p(r_0)} S \; dV + C_{11} \\ \leq & C V_p^{-1}(r_0) \int_{B_p(2r_0)} \rho^2 S \; dV + C_{11} \leq C_{12} \frac{V_p(2r_0)}{V_p(r_0)} + C_{11}. \end{aligned} \tag{5.27}$$

For each point $p \in \mathcal{T}_g$, let $f_p : \mathcal{T}_g \to \mathbb{C}^{3g-3}$ be the Bers' embedding map such that $f_p(p) = 0$ and $B_2 \subset f_p(\mathcal{T}_g) \subset B_6$ where $B_r \subset \mathbb{C}^{3g-3}$ is the open Euclidean ball of radius r. Since both metrics h and g are equivalent to the Ricci metric which is equivalent to the Euclidean metric on the unit Euclidean ball B_1, we know that $\frac{V_p(2r_0)}{V_p(r_0)}$ is uniformly bounded since both balls have Euclidean volume growth. Thus $f(p) \leq C_{13}$. Since Δu is bounded, we conclude that S is uniformly bounded.

Now we do the C^4 estimate. Let κ be a large positive constant. We first compute $\Delta^{'}[(S+\kappa)V]$. We have

$$\begin{aligned}\Delta^{'}[(S+\kappa)V] =& g^{i\overline{j}}\partial_i\partial_{\overline{j}}[(S+\kappa)V] = g^{i\overline{j}}\partial_{\overline{j}}[V\partial_i S + (S+\kappa)\partial_i V] \\ =& g^{i\overline{j}}\left[(S+\kappa)\partial_i\partial_{\overline{j}}V + V\partial_i\partial_{\overline{j}}S + \partial_i V\partial_{\overline{j}}S + \partial_i S\partial_{\overline{j}}V\right] \\ \geq& (S+\kappa)\Delta^{'}V + V\Delta^{'}S - 2\left|\nabla^{'}S\right|\left|\nabla^{'}V\right|. \end{aligned} \tag{5.28}$$

The computations of the first and second derivatives of V are very long. We only list the results here. For the first derivative of V, we have

$$\begin{aligned}\partial_\alpha V =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{j}p\overline{l}m} + u_{;i\overline{q}k\overline{n}}u_{;\overline{j}p\overline{l}m\alpha}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{\delta}\alpha}u_{;\gamma\overline{q}k\overline{n}}u_{;\overline{j}p\overline{l}m} + u_{;k\overline{\delta}\alpha}u_{;i\overline{q}\gamma\overline{n}}u_{;\overline{j}p\overline{l}m}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\gamma\overline{\delta}}\left[u_{;p\overline{\delta}\alpha}u_{;\overline{j}\gamma\overline{l}m}u_{;i\overline{q}k\overline{n}} + u_{;m\overline{\delta}\alpha}u_{;\overline{j}p\overline{l}\gamma}u_{;i\overline{q}k\overline{n}}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;i\overline{n}kp\alpha}u_{;\overline{j}m\overline{l}\overline{q}} + u_{;i\overline{n}kp}u_{;\overline{j}m\overline{l}\overline{q}\alpha}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{\delta}\alpha}u_{;\gamma\overline{n}kp}u_{;\overline{j}m\overline{l}\overline{q}} + u_{;k\overline{\delta}\alpha}u_{;i\overline{n}\gamma p}u_{;\overline{j}m\overline{l}\overline{q}}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\gamma\overline{\delta}}\left[u_{;p\overline{\delta}\alpha}u_{;i\overline{n}k\gamma}u_{;\overline{j}m\overline{l}\overline{q}} + u_{;m\overline{\delta}\alpha}u_{;\overline{j}\gamma\overline{l}\overline{q}}u_{;i\overline{n}kp}\right]. \end{aligned} \tag{5.29}$$

By differentiating the above formula we have

$$\Delta^{'}V = g^{\alpha\overline{\beta}}\partial_\alpha\partial_{\overline{\beta}}V = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \tag{5.30}$$

where

$$\begin{aligned}A_1 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{j}p\overline{l}m\overline{\beta}} + u_{;i\overline{q}k\overline{n}\overline{\beta}}u_{;\overline{j}p\overline{l}m\alpha}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}\left[u_{;i\overline{n}kp\alpha}u_{;\overline{j}m\overline{l}\overline{q}\overline{\beta}} + u_{;i\overline{n}kp\overline{\beta}}u_{;\overline{j}m\overline{l}\overline{q}\alpha}\right] \\ =& W, \end{aligned} \tag{5.31}$$

$$\begin{aligned}A_2 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}\left[u_{;i\overline{q}k\overline{n}\alpha\overline{\beta}}u_{;\overline{j}p\overline{l}m} + u_{;i\overline{q}k\overline{n}}u_{;\overline{j}p\overline{l}m\alpha\overline{\beta}}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}\left[u_{;i\overline{n}kp\alpha\overline{\beta}}u_{;\overline{j}m\overline{l}\overline{q}} + u_{;i\overline{n}kp}u_{;\overline{j}m\overline{l}\overline{q}\alpha\overline{\beta}}\right], \end{aligned} \tag{5.32}$$

$$\begin{aligned}A_3 =& - g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{\delta}k\overline{n}\alpha}u_{;\overline{q}\gamma\overline{\beta}}u_{;\overline{j}p\overline{l}m} + u_{;i\overline{q}k\overline{\delta}\alpha}u_{;\overline{n}\gamma\overline{\beta}}u_{;\overline{j}p\overline{l}m}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{j}\gamma\overline{\beta}}u_{;\overline{\delta}p\overline{l}m} + u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{l}\gamma\overline{\beta}}u_{;\overline{j}p\overline{\delta}m}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{\delta}k\overline{n}}u_{;\overline{q}\gamma\overline{\beta}}u_{;\overline{j}p\overline{l}m\alpha} + u_{;i\overline{q}k\overline{\delta}}u_{;\overline{n}\gamma\overline{\beta}}u_{;\overline{j}p\overline{l}m\alpha}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}}u_{;\overline{j}\gamma\overline{\beta}}u_{;\overline{\delta}p\overline{l}m\alpha} + u_{;i\overline{q}k\overline{n}}u_{;\overline{l}\gamma\overline{\beta}}u_{;\overline{j}p\overline{\delta}m\alpha}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;\gamma\overline{q}k\overline{n}\overline{\beta}}u_{;i\overline{\delta}\alpha}u_{;\overline{j}p\overline{l}m} + u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;i\overline{\delta}\alpha}u_{;\gamma\overline{q}k\overline{n}}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}\gamma\overline{n}\overline{\beta}}u_{;k\overline{\delta}\alpha}u_{;\overline{j}p\overline{l}m} + u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;k\overline{\delta}\alpha}u_{;i\overline{q}\gamma\overline{n}}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\overline{\beta}}u_{;p\overline{\delta}\alpha}u_{;\overline{j}\gamma\overline{l}m} + u_{;\overline{j}\gamma\overline{l}m\overline{\beta}}u_{;p\overline{\delta}\alpha}u_{;i\overline{q}k\overline{n}}\right] \\ &- g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\overline{\beta}}u_{;m\overline{\delta}\alpha}u_{;\overline{j}p\overline{l}\gamma} + u_{;\overline{j}p\overline{l}\gamma\overline{\beta}}u_{;m\overline{\delta}\alpha}u_{;i\overline{q}k\overline{n}}\right], \end{aligned} \tag{5.33}$$

$$
\begin{aligned}
A_4 =& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\bar{\delta}kp\alpha} u_{;\overline{n}\gamma\overline{\beta}} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;i\overline{n}kp\alpha} u_{;\bar{j}\gamma\overline{\beta}} u_{;\bar{\delta}m\bar{l}\overline{q}} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\overline{n}kp\alpha} u_{;\bar{l}\gamma\overline{\beta}} u_{;\bar{j}m\bar{\delta}\overline{q}} + u_{;i\overline{n}kp\alpha} u_{;\overline{q}\gamma\overline{\beta}} u_{;\bar{j}m\bar{l}\bar{\delta}} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\bar{\delta}kp} u_{;\overline{n}\gamma\overline{\beta}} u_{;\bar{j}m\bar{l}\overline{q}\alpha} + u_{;i\overline{n}kp} u_{;\bar{j}\gamma\overline{\beta}} u_{;\bar{\delta}m\bar{l}\overline{q}\alpha} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\overline{n}kp} u_{;\bar{l}\gamma\overline{\beta}} u_{;\bar{j}m\bar{\delta}\overline{q}\alpha} + u_{;i\overline{n}kp} u_{;\overline{q}\gamma\overline{\beta}} u_{;\bar{j}m\bar{l}\bar{\delta}\alpha} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;\gamma\overline{n}kp\overline{\beta}} u_{;i\bar{\delta}\alpha} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;\bar{j}m\bar{l}\overline{q}\overline{\beta}} u_{;i\bar{\delta}\alpha} u_{;\gamma\overline{n}kp} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\overline{n}\gamma p\overline{\beta}} u_{;k\bar{\delta}\alpha} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;\bar{j}m\bar{l}\overline{q}\overline{\beta}} u_{;k\bar{\delta}\alpha} u_{;i\overline{n}\gamma p} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\overline{n}k\gamma\overline{\beta}} u_{;p\bar{\delta}\alpha} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;\bar{j}m\bar{l}\overline{q}\overline{\beta}} u_{;p\bar{\delta}\alpha} u_{;i\overline{n}k\gamma} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\overline{n}kp\overline{\beta}} u_{;m\bar{\delta}\alpha} u_{;\bar{j}\gamma\bar{l}\overline{q}} + u_{;\bar{j}\gamma\bar{l}\overline{q}\overline{\beta}} u_{;m\bar{\delta}\alpha} u_{;i\overline{n}kp} \right],
\end{aligned}
\tag{5.34}
$$

$$
\begin{aligned}
A_5 =& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\bar{\delta}\alpha\overline{\beta}} u_{;\gamma\overline{q}k\overline{n}} u_{;\bar{j}p\bar{l}m} + u_{;k\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{q}\gamma\overline{n}} u_{;\bar{j}p\bar{l}m} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;p\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}\gamma\bar{l}m} + u_{;m\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}p\bar{l}\gamma} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;i\bar{\delta}\alpha\overline{\beta}} u_{;\gamma\overline{n}kp} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;k\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{n}\gamma p} u_{;\bar{j}m\bar{l}\overline{q}} \right] \\
& - g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} \left[u_{;p\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{n}k\gamma} u_{;\bar{j}m\bar{l}\overline{q}} + u_{;m\bar{\delta}\alpha\overline{\beta}} u_{;i\overline{n}kp} u_{;\bar{j}\gamma\bar{l}\overline{q}} \right],
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
A_6 =& g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;i\bar{t}\alpha} u_{;\bar{\delta}s\overline{\beta}} u_{;\gamma\overline{q}k\overline{n}} u_{;\bar{j}p\bar{l}m} + u_{;i\bar{\delta}\alpha} u_{;\overline{q}s\overline{\beta}} u_{;\gamma\bar{t}k\overline{n}} u_{;\bar{j}p\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;i\bar{\delta}\alpha} u_{;\overline{n}s\overline{\beta}} u_{;\gamma\overline{q}k\bar{t}} u_{;\bar{j}p\bar{l}m} + u_{;i\bar{\delta}\alpha} u_{;\bar{j}s\overline{\beta}} u_{;\gamma\overline{q}k\overline{n}} u_{;\bar{t}p\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;i\bar{\delta}\alpha} u_{;\bar{l}s\overline{\beta}} u_{;\gamma\overline{q}k\overline{n}} u_{;\bar{j}p\bar{t}m} + u_{;k\bar{t}\alpha} u_{;\bar{\delta}s\overline{\beta}} u_{;i\overline{q}\gamma\overline{n}} u_{;\bar{j}p\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;k\bar{\delta}\alpha} u_{;\overline{q}s\overline{\beta}} u_{;i\bar{t}\gamma\overline{n}} u_{;\bar{j}p\bar{l}m} + u_{;k\bar{\delta}\alpha} u_{;\overline{n}s\overline{\beta}} u_{;i\overline{q}\gamma\bar{t}} u_{;\bar{j}p\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;k\bar{\delta}\alpha} u_{;\bar{j}s\overline{\beta}} u_{;i\overline{q}\gamma\overline{n}} u_{;\bar{t}p\bar{l}m} + u_{;k\bar{\delta}\alpha} u_{;\bar{l}s\overline{\beta}} u_{;i\overline{q}\gamma\overline{n}} u_{;\bar{j}p\bar{t}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;p\bar{t}\alpha} u_{;\bar{\delta}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}\gamma\bar{l}m} + u_{;p\bar{\delta}\alpha} u_{;\bar{j}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{t}\gamma\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;p\bar{\delta}\alpha} u_{;\bar{l}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}\gamma\bar{t}m} + u_{;p\bar{\delta}\alpha} u_{;\overline{q}s\overline{\beta}} u_{;i\bar{t}k\overline{n}} u_{;\bar{j}\gamma\bar{l}m} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;p\bar{\delta}\alpha} u_{;\overline{n}s\overline{\beta}} u_{;i\overline{q}k\bar{t}} u_{;\bar{j}\gamma\bar{l}m} + u_{;m\bar{t}\alpha} u_{;\bar{\delta}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}p\bar{l}\gamma} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;m\bar{\delta}\alpha} u_{;\bar{j}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{t}p\bar{l}\gamma} + u_{;m\bar{\delta}\alpha} u_{;\bar{l}s\overline{\beta}} u_{;i\overline{q}k\overline{n}} u_{;\bar{j}p\bar{t}\gamma} \right] \\
& + g^{i\bar{j}} g^{k\bar{l}} g^{p\overline{q}} g^{m\overline{n}} g^{\alpha\overline{\beta}} g^{\gamma\bar{\delta}} g^{s\bar{t}} \left[u_{;m\bar{\delta}\alpha} u_{;\overline{q}s\overline{\beta}} u_{;i\bar{t}k\overline{n}} u_{;\bar{t}j\bar{l}\gamma} + u_{;m\bar{\delta}\alpha} u_{;\overline{n}s\overline{\beta}} u_{;i\overline{q}k\bar{t}} u_{;\bar{j}p\bar{l}\gamma} \right]
\end{aligned}
\tag{5.36}
$$

and

$$
\begin{aligned}
A_7 =& g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;i\bar{t}\alpha}u_{;\bar{\delta}s\bar{\beta}}u_{;\gamma\bar{n}kp}u_{;\bar{j}m\bar{l}\bar{q}} + u_{;i\bar{\delta}\alpha}u_{;\bar{n}s\bar{\beta}}u_{;\gamma\bar{t}kp}u_{;\bar{j}m\bar{l}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;i\bar{\delta}\alpha}u_{;\bar{j}s\bar{\beta}}u_{;\gamma\bar{n}kp}u_{;\bar{t}m\bar{l}\bar{q}} + u_{;i\bar{\delta}\alpha}u_{;\bar{l}s\bar{\beta}}u_{;\gamma\bar{n}kp}u_{;\bar{j}m\bar{t}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;i\bar{\delta}\alpha}u_{;\bar{q}s\bar{\beta}}u_{;\gamma\bar{n}kp}u_{;\bar{j}m\bar{l}\bar{t}} + u_{;k\bar{t}\alpha}u_{;\bar{\delta}s\bar{\beta}}u_{;i\bar{n}\gamma p}u_{;\bar{j}m\bar{l}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;k\bar{\delta}\alpha}u_{;\bar{n}s\bar{\beta}}u_{;i\bar{t}\gamma p}u_{;\bar{j}m\bar{l}\bar{q}} + u_{;k\bar{\delta}\alpha}u_{;\bar{j}s\bar{\beta}}u_{;i\bar{n}\gamma p}u_{;\bar{t}m\bar{l}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;k\bar{\delta}\alpha}u_{;\bar{l}s\bar{\beta}}u_{;i\bar{n}\gamma p}u_{;\bar{j}m\bar{t}\bar{q}} + u_{;k\bar{\delta}\alpha}u_{;\bar{q}s\bar{\beta}}u_{;i\bar{n}\gamma p}u_{;\bar{j}m\bar{l}\bar{t}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;p\bar{t}\alpha}u_{;\bar{\delta}s\bar{\beta}}u_{;i\bar{n}k\gamma}u_{;\bar{j}m\bar{l}\bar{q}} + u_{;p\bar{\delta}\alpha}u_{;\bar{n}s\bar{\beta}}u_{;i\bar{t}k\gamma}u_{;\bar{j}m\bar{l}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;p\bar{\delta}\alpha}u_{;\bar{j}s\bar{\beta}}u_{;i\bar{n}k\gamma}u_{;\bar{t}m\bar{l}\bar{q}} + u_{;p\bar{\delta}\alpha}u_{;\bar{l}s\bar{\beta}}u_{;i\bar{n}k\gamma}u_{;\bar{j}m\bar{t}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;p\bar{\delta}\alpha}u_{;\bar{q}s\bar{\beta}}u_{;i\bar{n}k\gamma}u_{;\bar{j}m\bar{l}\bar{t}} + u_{;m\bar{t}\alpha}u_{;\bar{\delta}s\bar{\beta}}u_{;i\bar{n}kp}u_{;\bar{j}\gamma\bar{l}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;m\bar{\delta}\alpha}u_{;\bar{j}s\bar{\beta}}u_{;i\bar{n}kp}u_{;\bar{t}\gamma\bar{l}\bar{q}} + u_{;m\bar{\delta}\alpha}u_{;\bar{l}s\bar{\beta}}u_{;i\bar{n}kp}u_{;\bar{j}\gamma\bar{t}\bar{q}}\right] \\
&+ g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}g^{s\bar{t}}\left[u_{;m\bar{\delta}\alpha}u_{;\bar{q}s\bar{\beta}}u_{;i\bar{n}kp}u_{;\bar{j}\gamma\bar{l}\bar{t}} + u_{;m\bar{\delta}\alpha}u_{;\bar{n}s\bar{\beta}}u_{;i\bar{t}kp}u_{;\bar{j}\gamma\bar{l}\bar{q}}\right].
\end{aligned} \tag{5.37}
$$

Now we estimate each A_i in the sum of $\Delta' V$. Since we have C^3 estimate, that is, S is bounded, by diagonalizing, we know that each term in the sum A_6 and A_7 is bounded by a constant multiple of V. So we have

$$|A_6| + |A_7| \leq C_{14}V. \tag{5.38}$$

Now we estimate terms in A_5. We have

$$u_{;i\bar{\delta}\alpha\bar{\beta}} = u_{;\alpha\bar{\beta}i\bar{\delta}} + (u_{;i\bar{\delta}\alpha\bar{\beta}} - u_{;\alpha\bar{\beta}i\bar{\delta}}) = u_{;\alpha\bar{\beta}i\bar{\delta}} + h^{s\bar{t}}(u_{s\bar{\beta}}R_{\alpha\bar{t}i\bar{\delta}} - u_{s\bar{\delta}}R_{i\bar{t}\alpha\bar{\beta}}).$$

By using equation (5.17) we have

$$
\begin{aligned}
g^{\alpha\bar{\beta}}u_{;i\bar{\delta}\alpha\bar{\beta}} =& g^{\alpha\bar{\beta}}u_{;\alpha\bar{\beta}i\bar{\delta}} + g^{\alpha\bar{\beta}}h^{s\bar{t}}(u_{s\bar{\beta}}R_{\alpha\bar{t}i\bar{\delta}} - u_{s\bar{\delta}}R_{i\bar{t}\alpha\bar{\beta}}) \\
=& \mathcal{F}_{i\bar{\delta}} + g^{\alpha\bar{\beta}}g^{s\bar{t}}u_{;\alpha\bar{t}i}u_{;\bar{\beta}s\bar{\delta}} + g^{\alpha\bar{\beta}}h^{s\bar{t}}(u_{s\bar{\beta}}R_{\alpha\bar{t}i\bar{\delta}} - u_{s\bar{\delta}}R_{i\bar{t}\alpha\bar{\beta}}).
\end{aligned}
$$

Since the curvature of the metric h is bounded and we have C^2 and C^3 estimate, we know that

$$\left| -g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}g^{m\bar{n}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}u_{;i\bar{\delta}\alpha\bar{\beta}}u_{;\gamma\bar{q}k\bar{n}}u_{;\bar{j}p\bar{l}m} \right| \leq CV.$$

Similarly we can compute other terms in the sum A_5. So we have

$$|A_5| \leq C_{15}V. \tag{5.39}$$

By combining formulas (5.30), (5.38) and (5.39) we have

$$\Delta' V \geq A_1 + A_2 + A_3 + A_4 - C_{16}V. \tag{5.40}$$

Now we deal with terms in A_2. By differentiating formula (5.18) in a holomorphic direction or an anti-holomorphic direction we have

$$\begin{aligned}
g^{i\bar{j}}u_{;i\bar{j}k\bar{l}\alpha\overline{\beta}} =& g^{i\bar{j}}g^{p\overline{q}}\left[u_{;i\overline{q}k\bar{l}\alpha}u_{;\bar{j}p\overline{\beta}} + u_{;i\overline{q}k\alpha\overline{\beta}}u_{;\bar{j}p\bar{l}} + u_{;p\bar{j}k\bar{l}\overline{\beta}}u_{;i\overline{q}\alpha} + u_{;\bar{j}p\bar{l}\alpha\overline{\beta}}u_{;i\overline{q}k}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}\left[u_{;p\bar{j}k\bar{l}}u_{;i\overline{q}\alpha\overline{\beta}} + u_{;\bar{j}p\bar{l}\alpha}u_{;i\overline{q}k\overline{\beta}} + u_{;i\overline{q}k\alpha}u_{;\bar{j}p\bar{l}\overline{\beta}}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;p\overline{n}k\bar{l}}u_{;\bar{j}m\overline{\beta}}u_{;i\overline{q}\alpha} + u_{;p\bar{j}k\bar{l}}u_{;\overline{q}m\overline{\beta}}u_{;i\overline{n}\alpha} + u_{;\overline{n}p\bar{l}\alpha}u_{;\bar{j}m\overline{\beta}}u_{;i\overline{q}k}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;\bar{j}p\bar{l}\alpha}u_{;\overline{q}m\overline{\beta}}u_{;i\overline{n}k} + u_{;i\overline{n}k\alpha}u_{;\overline{q}m\overline{\beta}}u_{;\bar{j}p\bar{l}} + u_{;i\overline{q}k\alpha}u_{;\bar{j}m\overline{\beta}}u_{;\overline{n}p\bar{l}}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;\bar{j}m\bar{l}\overline{\beta}}u_{;p\overline{n}\alpha}u_{;i\overline{q}k} + u_{;\bar{j}p\bar{l}\overline{\beta}}u_{;i\overline{n}\alpha}u_{;m\overline{q}k} + u_{;p\overline{n}\alpha\overline{\beta}}u_{;\bar{j}m\bar{l}}u_{;i\overline{q}k}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;i\overline{q}k\overline{\beta}}u_{;\bar{j}m\bar{l}}u_{;p\overline{n}\alpha} + u_{;m\overline{q}k\overline{\beta}}u_{;\bar{j}p\bar{l}}u_{;i\overline{n}\alpha} + u_{;i\overline{n}\alpha\overline{\beta}}u_{;\bar{j}p\bar{l}}u_{;m\overline{q}k}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;i\overline{q}k}u_{;\bar{j}s\overline{\beta}}u_{;p\overline{n}\alpha}u_{;\bar{t}m\bar{l}} + u_{;i\overline{q}k}u_{;\overline{n}s\overline{\beta}}u_{;p\bar{t}\alpha}u_{;\bar{j}m\bar{l}}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;i\bar{t}k}u_{;\overline{q}s\overline{\beta}}u_{;p\overline{n}\alpha}u_{;\bar{j}m\bar{l}} + u_{;m\bar{t}k}u_{;\overline{q}s\overline{\beta}}u_{;i\overline{n}\alpha}u_{;\bar{j}p\bar{l}}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;m\overline{q}k}u_{;\overline{n}s\overline{\beta}}u_{;i\bar{t}\alpha}u_{;\bar{j}p\bar{l}} + u_{;m\overline{q}k}u_{;\bar{j}s\overline{\beta}}u_{;i\overline{n}\alpha}u_{;\bar{t}p\bar{l}}\right] \\
&+ u_{;k\bar{l}\alpha\overline{\beta}} + F_{;k\bar{l}\alpha\overline{\beta}}
\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}
g^{i\bar{j}}u_{;i\bar{j}k\bar{l}\alpha\gamma} =& g^{i\bar{j}}g^{p\overline{q}}\left[u_{;p\bar{j}k\bar{l}\alpha}u_{;i\overline{q}\gamma} + u_{;p\bar{j}k\bar{l}\gamma}u_{;i\overline{q}\alpha} + u_{;i\overline{q}k\alpha\gamma}u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\alpha\gamma}u_{;i\overline{q}k}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}\left[u_{;p\bar{j}k\bar{l}}u_{;i\overline{q}\alpha\gamma} + u_{;\bar{j}p\bar{l}\alpha}u_{;i\overline{q}k\gamma} + u_{;i\overline{q}k\alpha}u_{;\bar{j}p\bar{l}\gamma}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;m\bar{j}k\bar{l}}u_{;p\overline{n}\gamma}u_{;i\overline{q}\alpha} + u_{;p\bar{j}k\bar{l}}u_{;i\overline{n}\gamma}u_{;m\overline{q}\alpha} + u_{;\bar{j}m\bar{l}\alpha}u_{;p\overline{n}\gamma}u_{;i\overline{q}k}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;\bar{j}p\bar{l}\alpha}u_{;i\overline{n}\gamma}u_{;m\overline{q}k} + u_{;m\overline{q}k\alpha}u_{;i\overline{n}\gamma}u_{;\bar{j}p\bar{l}} + u_{;i\overline{q}k\alpha}u_{;p\overline{n}\gamma}u_{;\bar{j}m\bar{l}}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;i\overline{q}k\gamma}u_{;p\overline{n}\alpha}u_{;\bar{j}m\bar{l}} + u_{;p\overline{n}\alpha\gamma}u_{;i\overline{q}k}u_{;\bar{j}m\bar{l}} + u_{;\bar{j}m\bar{l}\gamma}u_{;i\overline{q}k}u_{;p\overline{n}\alpha}\right] \\
&- g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}\left[u_{;m\overline{q}k\gamma}u_{;i\overline{n}\alpha}u_{;\bar{j}p\bar{l}} + u_{;i\overline{n}\alpha\gamma}u_{;m\overline{q}k}u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\gamma}u_{;m\overline{q}k}u_{;i\overline{n}\alpha}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;i\bar{t}\gamma}u_{;s\overline{q}k}u_{;p\overline{n}\alpha}u_{;\bar{j}m\bar{l}} + u_{;p\bar{t}\gamma}u_{;i\overline{q}k}u_{;s\overline{n}\alpha}u_{;\bar{j}m\bar{l}}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;m\bar{t}\gamma}u_{;i\overline{q}k}u_{;p\overline{n}\alpha}u_{;\bar{j}s\bar{l}} + u_{;m\bar{t}\gamma}u_{;s\overline{q}k}u_{;i\overline{n}\alpha}u_{;\bar{j}p\bar{l}}\right] \\
&+ g^{i\bar{j}}g^{p\overline{q}}g^{m\overline{n}}g^{s\bar{t}}\left[u_{;i\bar{t}\gamma}u_{;m\overline{q}k}u_{;s\overline{n}\alpha}u_{;\bar{j}p\bar{l}} + u_{;p\bar{t}\gamma}u_{;m\overline{q}k}u_{;i\overline{n}\alpha}u_{;\bar{j}s\bar{l}}\right] \\
&+ u_{;k\bar{l}\alpha\gamma} + F_{;k\bar{l}\alpha\gamma}.
\end{aligned} \tag{5.42}$$

By using a similar computation as in [19] we have

$$u_{;i\overline{q}k\overline{n}\alpha\overline{\beta}} \overset{4}{\simeq} u_{;\alpha\overline{\beta}i\overline{q}k\overline{n}}, \tag{5.43}$$

$$u_{;\bar{j}p\bar{l}m\alpha\overline{\beta}} \overset{4}{\simeq} u_{;\overline{\beta}\alpha\bar{j}p\bar{l}m}, \tag{5.44}$$

$$u_{;i\overline{n}kp\alpha\overline{\beta}} \overset{4}{\simeq} u_{;\alpha\overline{\beta}i\overline{n}kp} \tag{5.45}$$

and

$$u_{;\bar{j}m\bar{l}\overline{q}\alpha\overline{\beta}} \overset{4}{\simeq} u_{;\overline{\beta}\alpha\bar{j}m\bar{l}\overline{q}} \tag{5.46}$$

which imply that

$$g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;i\overline{q}k\overline{n}\alpha\overline{\beta}}u_{;\overline{j}p\overline{l}m} \overset{4}{\cong} g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\alpha\overline{\beta}i\overline{q}k\overline{n}}u_{;\overline{j}p\overline{l}m}, \tag{5.47}$$

$$g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{j}p\overline{l}m\alpha\overline{\beta}}u_{;i\overline{q}k\overline{n}} \overset{4}{\cong} g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{\beta}\alpha\overline{j}p\overline{l}m}u_{;i\overline{q}k\overline{n}}, \tag{5.48}$$

$$g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;i\overline{n}kp\alpha\overline{\beta}}u_{;\overline{j}m\overline{l}\overline{q}} \overset{4}{\cong} g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\alpha\overline{\beta}i\overline{n}kp}u_{;\overline{j}m\overline{l}\overline{q}} \tag{5.49}$$

and

$$g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{j}m\overline{l}\overline{q}\alpha\overline{\beta}}u_{;i\overline{n}kp} \overset{4}{\cong} g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{\beta}\alpha\overline{j}m\overline{l}\overline{q}}u_{;i\overline{n}kp}. \tag{5.50}$$

By using equations (5.41), (5.42) and their conjugations, we have

$$\left|g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\alpha\overline{\beta}i\overline{q}k\overline{n}}u_{;\overline{j}p\overline{l}m} - T_1\right| \le C_{17}V^{\frac{3}{2}} + C_{18}V + C_{19}V^{\frac{1}{2}}, \tag{5.51}$$

$$\left|g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{\beta}\alpha\overline{j}p\overline{l}m}u_{;i\overline{q}k\overline{n}} - T_2\right| \le C_{17}V^{\frac{3}{2}} + C_{18}V + C_{19}V^{\frac{1}{2}}, \tag{5.52}$$

$$\left|g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\alpha\overline{\beta}i\overline{n}kp}u_{;\overline{j}m\overline{l}\overline{q}} - T_3\right| \le C_{17}V^{\frac{3}{2}} + C_{18}V + C_{19}V^{\frac{1}{2}} \tag{5.53}$$

and

$$\left|g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}u_{;\overline{\beta}\alpha\overline{j}m\overline{l}\overline{q}}u_{;i\overline{n}kp} - T_4\right| \le C_{17}V^{\frac{3}{2}} + C_{18}V + C_{19}V^{\frac{1}{2}} \tag{5.54}$$

where

$$\begin{aligned} T_1 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{j}p\overline{\delta}}u_{;\overline{l}m\overline{\beta}\gamma} + u_{;i\overline{n}kp\overline{\beta}}u_{;\overline{j}m\overline{\delta}}u_{;\overline{l}\gamma\overline{q}\alpha}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\overline{\beta}}u_{;p\overline{j}\gamma}u_{;\overline{l}m\overline{\delta}\alpha} + u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;i\overline{q}\gamma}u_{;\overline{\delta}k\overline{n}\alpha}\right], \end{aligned} \tag{5.55}$$

$$\begin{aligned} T_2 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;i\overline{q}\gamma}u_{;k\overline{n}\alpha\overline{\delta}} + u_{;\overline{j}m\overline{l}\overline{q}\alpha}u_{;i\overline{n}\gamma}u_{;k\overline{\delta}p\overline{\beta}}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;\overline{j}p\overline{l}m\alpha}u_{;\overline{q}i\overline{\delta}}u_{;k\overline{n}\gamma\overline{\beta}} + u_{;i\overline{q}k\overline{n}\alpha}u_{;\overline{j}p\overline{\delta}}u_{;\gamma\overline{l}m\overline{\beta}}\right], \end{aligned} \tag{5.56}$$

$$\begin{aligned} T_3 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;p\overline{j}\gamma}u_{;\overline{l}m\overline{\beta}\overline{\delta}} + u_{;i\overline{n}kp\alpha}u_{;\overline{j}m\overline{\delta}}u_{;\overline{l}\gamma\overline{q}\overline{\beta}}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;i\overline{q}k\overline{n}\alpha}u_{;p\overline{j}\gamma}u_{;\overline{l}m\overline{\delta}\overline{\beta}} + u_{;\overline{j}p\overline{l}m\alpha}u_{;i\overline{q}\gamma}u_{;\overline{\delta}k\overline{n}\overline{\beta}}\right] \end{aligned} \tag{5.57}$$

and

$$\begin{aligned} T_4 =& g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;\overline{q}i\overline{\delta}}u_{;k\overline{n}\alpha\gamma} + u_{;\overline{j}m\overline{l}\overline{q}\overline{\beta}}u_{;i\overline{n}\gamma}u_{;k\overline{\delta}p\alpha}\right] \\ &+ g^{i\overline{j}}g^{k\overline{l}}g^{p\overline{q}}g^{m\overline{n}}g^{\alpha\overline{\beta}}g^{\gamma\overline{\delta}}\left[u_{;\overline{j}p\overline{l}m\overline{\beta}}u_{;\overline{q}i\overline{\delta}}u_{;k\overline{n}\gamma\alpha} + u_{;i\overline{q}k\overline{n}\overline{\beta}}u_{;\overline{j}p\overline{\delta}}u_{;\gamma\overline{l}m\alpha}\right]. \end{aligned} \tag{5.58}$$

Now we choose local coordinates such that $g_{i\overline{j}} = \delta_{ij}$. By combining formulas (5.40), (5.33), (5.34) and (5.47)−(5.58) we have

$$\begin{aligned} \Delta^{'}V \ge& \sum_{i,p,k,m,\alpha}\left[|A_{ipkm\alpha}|^2 + |B_{ipkm\alpha}|^2 + |C_{ipkm\alpha}|^2 + |D_{ipkm\alpha}|^2\right] \\ &- C_{20}V^{\frac{3}{2}} - C_{21}V - C_{22}V^{\frac{1}{2}} \\ =&\widetilde{W} - C_{20}V^{\frac{3}{2}} - C_{21}V - C_{22}V^{\frac{1}{2}}. \end{aligned} \tag{5.59}$$

Now we estimate $\left|\nabla' V\right|$. For each fixed α, by (5.29) we have

$$\partial_\alpha V = \sum_{i,p,k,m} \left[A_{ipkm\alpha} u_{;\bar{i}p\bar{k}m} + B_{ipkm\alpha} u_{;i\bar{p}k\overline{m}} + C_{ipkm\alpha} u_{;\bar{i}m\overline{k}\overline{p}} + D_{ipkm\alpha} u_{;i\overline{m}kp} \right] + X_\alpha \tag{5.60}$$

where

$$|X_\alpha| \le CV.$$

By using the Schwarz inequality, it is easy to see that

$$|\partial_\alpha V| \le \sqrt{2} V^{\frac{1}{2}} \widetilde{W}^{\frac{1}{2}} + CV$$

which implies

$$\left|\nabla' V\right| \le C_{23} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} + C_{24} V. \tag{5.61}$$

Now we estimate the derivatives of S. By using similar computation as above, we have

$$\begin{aligned} \partial_\alpha S =& g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \left(u_{;i\bar{q}k\alpha} u_{;\bar{j}p\bar{l}} + u_{;\bar{j}p\bar{l}\alpha} u_{;i\bar{q}k} \right) \\ &- g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} \left(u_{;m\bar{q}k} u_{;i\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}m} u_{;k\bar{n}\alpha} u_{;\bar{j}p\bar{l}} + u_{;i\bar{q}k} u_{;p\bar{n}\alpha} u_{;\bar{j}m\bar{l}} \right). \end{aligned} \tag{5.62}$$

Since S is bounded, we have

$$|\partial_\alpha S| \le CV^{\frac{1}{2}} + \widetilde{C}$$

which implies

$$\left|\nabla' S\right| \le C_{25} V^{\frac{1}{2}} + C_{26}. \tag{5.63}$$

By Yau's work in [19] we have

$$\Delta' S \ge V - C_{27} V^{\frac{1}{2}} - C_{28}. \tag{5.64}$$

From (5.59), we have

$$\begin{aligned} (S+\kappa)\Delta' V \ge& (S+\kappa) \left(\widetilde{W} - C_{20} V^{\frac{3}{2}} - C_{21} V - C_{22} V^{\frac{1}{2}} \right) \\ \ge& \kappa \widetilde{W} - C_{29} V^{\frac{3}{2}} - C_{30} V - C_{31} V^{\frac{1}{2}} \end{aligned} \tag{5.65}$$

where the constants C_{29}, C_{30} and C_{31} depend on κ. By (5.64), (5.61) and (5.64) we also have

$$V \Delta' S \ge V^2 - C_{27} V^{\frac{3}{2}} - C_{28} V \tag{5.66}$$

and

$$2 \left|\nabla' S\right| \left|\nabla' V\right| \le C_{32} \widetilde{W}^{\frac{1}{2}} V + C_{33} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} + C_{34} V^{\frac{3}{2}} + C_{35} V. \tag{5.67}$$

Combining (5.28), (5.65), (5.66) and (5.67) we have

$$\Delta' \left[(S+\kappa) V \right] \ge \kappa \widetilde{W} + V^2 - C_{32} \widetilde{W}^{\frac{1}{2}} V - C_{33} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} - C_{36} V^{\frac{3}{2}} - C_{37} V - C_{38} V^{\frac{1}{2}} \tag{5.68}$$

where constants C_{36}, C_{37} and C_{38} depend on κ. Now we fix a κ such that

$$\kappa \ge \max\{3C_{32}^2, 3C_{33}^2, 1\}.$$

We have

$$\frac{\kappa}{3} \widetilde{W} - C_{32} \widetilde{W}^{\frac{1}{2}} V + \frac{1}{4} V^2 \ge 0$$

and

$$\frac{\kappa}{3} \widetilde{W} - C_{33} \widetilde{W}^{\frac{1}{2}} V^{\frac{1}{2}} \ge -\frac{1}{4} V.$$

With this choice of κ, by formula (5.68) we have

$$\begin{aligned}\Delta'\left[(S+\kappa)V\right] \geq & \frac{\kappa}{3}\widetilde{W}+\frac{3}{4}V^2-C_{39}V^{\frac{3}{2}}-C_{40}V-C_{41}V^{\frac{1}{2}} \\ \geq & C_{42}\left[(S+\kappa)V\right]^2-C_{43}\left[(S+\kappa)V\right]^{\frac{3}{2}}-C_{44}\left[(S+\kappa)V\right]-C_{45}\left[(S+\kappa)V\right]^{\frac{1}{2}}\end{aligned} \tag{5.69}$$

since $S+\kappa\geq\kappa\geq 1$ and $S+\kappa$ is bounded from above uniformly.

By the work of Cheng and Yau in [1], we know inequality (5.69) implies that $(S+\kappa)V$ is bounded. This implies V is bounded since $V\leq(S+\kappa)V$. Thus we obtain the C^4 estimate. By formula (5.16) we know that the curvature of the Kähler-Einstein metric g is bounded. This finishes the proof of Theorem 5.1.

Now we briefly describe how to control the covariant derivatives of the curvature of the Kähler-Einstein metric.

Firstly, by differentiating equation (5.16), we see that the boundedness of the derivatives of the $P_{i\bar{j}k\bar{l}}$ is equivalent to the boundedness of the covariant derivatives of u with respect to the background metric. Furthermore, the derivatives involved are at least order 2 and were taken in at least one holomorphic direction and one anti-holomorphic direction.

To bound such k-th order derivatives of u, we form the quantity S_k such that $S_3=S$, $S_4=V$ and $S_5=W$. In general, if we fix normal coordinates with respect to the Kähler-Einstein metric at one point, then S_k is a sum of square of terms where each term is a covariant derivative of u and the derivative is described above. All terms are obtained in the following way:

For each covariant derivative of u whose square appeared in the sum S_{k-1}, we take covariant derivative of this term with respect to the background metric in z_α and $\overline{z}_\beta$ respectively. Then we obtain two terms whose square appear in the sum S_k. It is easy to see that S_k is a sum of 2^{n-3} squares of certain covariant derivatives of u where the derivatives are of the type described above.

It is clear that the covariant derivatives of the curvature of the Kähler-Einstein metric is bounded is equivalent to the fact that the quantities S_k is bounded.

We now estimate S_k inductively. Assume S_l is bounded for any $l\leq k-1$, we compute

$$\Delta'\left((S_{k-1}+\kappa)S_k\right)$$

where κ is a large constant. Similar to inequality (5.28) we have

$$\Delta'\left((S_{k-1}+\kappa)S_k\right)\geq(S_{k-1}+\kappa)\Delta' S_k+S_k\Delta' S_{k-1}-2\left|\nabla' S_{k-1}\right|\left|\nabla' S_k\right|.$$

In the above formula, the leading term of $\Delta' S_{k-1}$ is S_k as we did in formula (5.59) and the term $\nabla' S_{k-1}$ is of order $S_k^{\frac{1}{2}}$ as we know in formula (5.63). Similarly, the term $\nabla' S_k$ is of order $S_{k+1}^{\frac{1}{2}}S_k^{\frac{1}{2}}$ as we did in formula (5.61). When we compute $\Delta' S_k$, the leading term is S_{k+1}. However, there will be products of $(k+2)$-th order derivatives of u and k-th order derivatives of u. We can reduce the $(k+2)$-th order derivatives of u by using the Monge-Ampére equation as we did in formulas (5.41)-(5.58). That is, by differentiating equation (5.17) successively and by switching the order of derivatives, we see that these products are of order at most $S_{k+1}^{\frac{1}{2}}S_k^{\frac{3}{2}}$.

By using similar argument as above, finally we can derive an inequality of form (5.69) when κ is large enough. By using Cheng-Yau's work, we conclude that S_k is bounded. The computation is very long but straightforward. We omit it here for simplicity .

□

As a direct corollary, we have

Corollary 5.1. *The injectivity radius of the Kähler-Einstein metric on the Teichmüller space is bounded from below. Thus the Teichmüller space equipped with this metric has bounded geometry.*

This corollary can be proved in the same way as Corollary 3.2 by using the above theorem.

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